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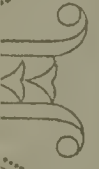


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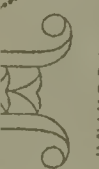
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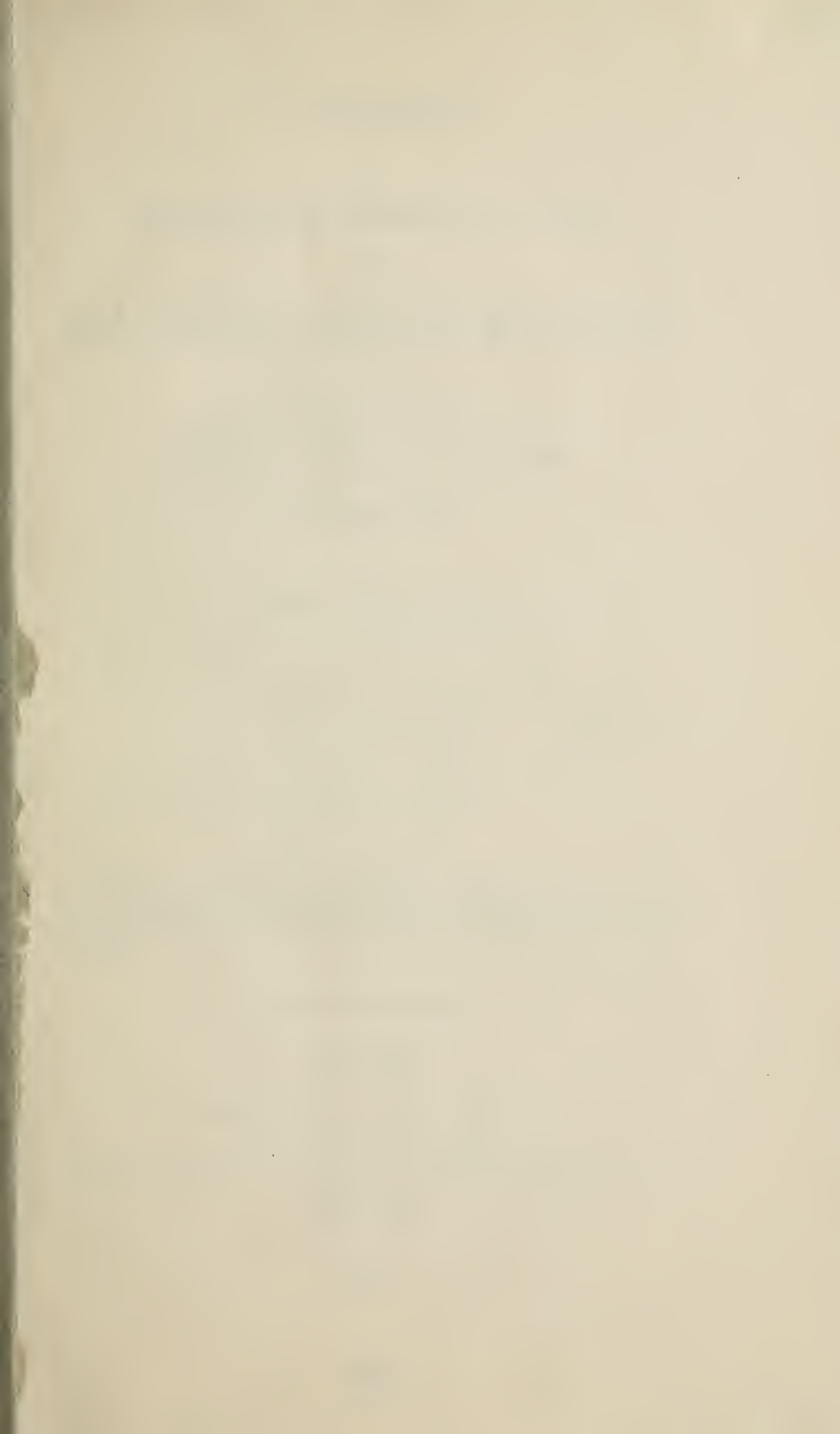


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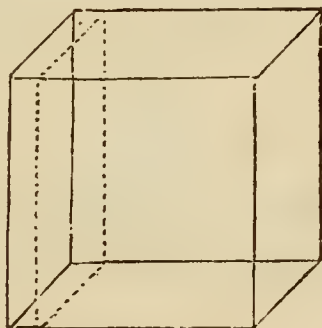
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II

A METHOD
OF
APPROXIMATING TOWARDS THE ROOTS
OF
CUBIC EQUATIONS
BELONGING TO THE
IRREDUCIBLE CASE.

BY
JAMES LOCKHART.



“Methods of resolving equations by approximation ought to be considered as of the highest utility, and as being absolutely necessary to the completion of the doctrine of the resolution of algebraic equations, which is the most important branch of the science of Algebra.”

MASERES.

LONDON:

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1813.

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THIS WORK

IS RESPECTFULLY DEDICATED

(BY PERMISSION)

TO THE

REV. ISAAC MILNER, D.D.

PRESIDENT OF QUEEN'S COLLEGE,

AND

LUCASIAN PROFESSOR OF MATHEMATICS IN THE
UNIVERSITY OF CAMBRIDGE,

BY HIS OBEDIENT AND HUMBLE SERVANT,

JAMES LOCKHART.

LONDON, *May* 1, 1813.

QA 218
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INTRODUCTION.

ALL the known methods of approximating towards the roots of equations of the third and superior orders, are attended with great labour and uncertainty. Previously to their application to a given equation, one or two figures of the root must be found by trial, or an assumption made, to serve as a basis for subsequent results. If two correct figures, found by trial, are employed as a first near value of the root, the subsequent result is generally in excess, and frequently several steps of the same method of approximation are requisite to correct it: if the root x of the equation $x^3 - 6x = 1$ be approximated to by Raphson's method, and the figures 2.5 taken as a first near value, such excess and slowness of approximation will be found to take place. If an assumption of figures is made, great judgment, founded on long experience, is necessary, otherwise the approximation will be very tedious, and oftentimes not more than two correct figures will be obtained after three applications of any of the methods. Results obtained by De Lagny's method are, in general, the most correct, but the operation is tedious, and the approximation cannot be absolutely confided in. The convergency necessary to the resolution of equations of one order is frequently inapplicable to others, and even in equations of the same order the same convergency is insufficient. Hence originate the unequal errors in excess and defect which persons skilled in approximation continually experience. Nor is this the only obstacle to a speedy solution; for, when two figures of the root have been assumed or found by trial, and three additional and supposed correct figures have been obtained by a first step of approximation, there is no mode of proving the work but by raising the figures to the given power, and making the necessary additions or subtractions; and even then doubt will frequently remain. The most expeditious

and judicious approximator can seldom obtain nine correct figures of the root in less than two hours, and he will in many cases be employed three or four hours about a single equation. If he resorts to the method of series, the length of the operation is intolerable.

The object of this work is, the resolution of cubic equations only, including all those belonging to the irreducible case, and rather more than one fourth of those, the roots of which are accurately discoverable by Tartaglia's rule. The method which it contains is, however, more peculiarly applicable to the resolution of those equations to which that rule cannot be accommodated. It commences with giving a first near value of the root without assumption or trial, and possesses a due convergency; the results are always accurate, and the operation is so simple, that a boy capable of extracting the square root can perform it with ease. Sometimes all the necessary figures are found by one step of the approximation, and in no case whatever will it require more than half an hour to obtain eight figures of the root of the given equation.

Before the author ventured to publish his method, he communicated it to some gentlemen profound in mathematical science, who gave him encouragement to hope, that it would meet the public favour; and should that hope be realized, he will endeavour to make the same principle suit equations of the fifth order. He has not given all the demonstrations which might have been expected: they would have increased the work to an inconvenient size; or, by excluding many examples, rendered it unfit for that purpose which he trusts will be its destiny, namely, that of giving the student a taste for approximation. The astronomical purposes to which the method may be adapted, need not be dilated on.

The greater part of the examples have been selected in order to resist the method as much as possible. Some of them include all that is difficult in the irreducible case; and the author assures his readers, that he is not acquainted with a single defect that attaches to this method.

METHOD OF APPROXIMATION.

THE irreducible case is sufficiently comprehended under the two equations $x^3 - bx = c$ and $bx - x^3 = c$,

$\frac{b^3}{c^2}$ being greater than 6.75 in each equation.

The former equation has only one root; the latter equation has two roots, the sum of which is equal to the root of the former equation. A general and finite expression for the values of the roots, is not known, and all that we can do, in innumerable cases, is to find approximating values.

The following rules are necessary to a right understanding of the method contained in this work, and serve to facilitate the requisite operations.

AUXILIARY RULES.

RULE I.

To find the difference of two cubes, their cube roots being given.

Make their difference equal to three times the difference of their cube roots multiplied into the square of the arithmetical mean of their cube roots, and twice the cube of half the difference of their cube roots: for,

$$x^3 - y^3 = \overline{3x - 3y} \times \left(\frac{x+y}{2}\right)^2 + 2 \left(\frac{x-y}{2}\right)^3$$

$$\text{or, } x^3 - y^3 = \frac{3d}{4} \times \overline{x+y}^2 + \frac{d^3}{4}$$

d being equal to $x - y$.

Ex.

The cube roots 476 and 472 being given, to find the difference of their cubes.

$$\overline{476+472}^2 = 898704$$

$$\frac{12}{4} = \underline{\quad 3 \quad}$$

$$2696112$$

$$\frac{64}{4} = \underline{\quad 16 \quad}$$

2696128 is the difference of the cubes.

This rule is applicable when ten or eleven figures have been found by the method, should there be a doubt respecting the accuracy of the last figure. The approximator will then be relieved from the necessity of cubing two high numbers. Cubing need not be resorted to at that state of the approximation, because a more simple method of proving the work is introduced; but it may be preferred from habit.

By means of this rule the differences of very high powers of numbers may be obtained with comparatively little trouble.

Ex.

Let $x^9 - y^3$ be required.

$$x^9 - y^3 = \overline{3x^3 - 3y} \times \left(\frac{x^3 + y}{2}\right)^2 + 2\left(\frac{x^3 - y}{2}\right)^3$$

This rule may be of some use in the method of increments; and it is obvious, that the difference of any two cubes may be geometrically found by means thereof.

RULE II.

Three numbers in arithmetical progression being given, to determine whether their product is connected or not with the irreducible case, the middle number being considered as the root of the equation $x^3 - bx = c$.

The progression may be represented thus:

$$\overline{x - \sqrt{b}} \times x \times \overline{x + \sqrt{b}} = x^3 - bx,$$

which may or may not be connected with the case in question.

The rule in respect of numbers is this:—Multiply the middle number by $1 - \sqrt{.75} = .13397459 +$: if the product is greater than the first

term, the product of the three terms is connected with the irreducible case, otherwise it is not: for $x \times 1 - \sqrt{.75} = x - \sqrt{.75}x$; and if this quantity is equal to the first term, the product of the three terms becomes $x^3 - .75x^3 = \frac{x^3}{4}$, or $x - \sqrt{.75}x \times x \times x + \sqrt{.75}x = \frac{x^3}{4}$: but this is the limit of the two cases, and the rule is therefore correct.

Ex.

Let 1, 10, 19 be the three terms.

$1 - \sqrt{.75} \times 10$ is greater than the first term; therefore the product of the three terms is connected with the irreducible case.

The resulting equation is $x^3 - 81x = 190 =$ their product :

$$\text{for } 10 - \sqrt{81} \times 10 \times 10 + \sqrt{81} = 190,$$

$$\text{and } x - \sqrt{81} \times x \times x + \sqrt{81} = x^3 - 81x.$$

Let three other terms be 2, 10, 18.

$1 - \sqrt{.75} \times 10$ is less than the first term; therefore the product of the three terms is not connected with the irreducible case.

The resulting equation is $x^3 - 64x = 360 =$ their product :

$$\text{for } 10 - \sqrt{64} \times 10 \times 10 + \sqrt{64} = 360,$$

$$\text{and } x - \sqrt{64} \times x \times x + \sqrt{64} = x^3 - 64x.$$

We may pass rapidly over series of this kind, and determine, almost by inspection, whether they are connected or not with the subject before us. The application of this rule will be shewn hereafter.

RULE III.

To determine whether the equations $x^3 - bx = c$, and $bx - x^3 = c$, are connected with the irreducible case or not.

Find the value of $\frac{b^3}{c^2}$

If $\frac{b^3}{c^2}$ is greater than 6.75, the equations are connected with the case; otherwise not. We know, that if, in the application of Tartaglia's rule to these equations, $\frac{b^3}{27}$ cannot be subtracted from $\frac{c^2}{4}$, the equations are irreducible, and that when these quantities are equal to each other, $x^3 - bx = \frac{x^3}{4}$: but this equation is the boun-

dary of the case which is reducible, and of that which is not reducible. If, then, $\frac{b^3}{27} = \frac{c^2}{4}$, it results that $\frac{b^3}{c^2} = 6.75$; and consequently that the equation is irreducible if $\frac{b^3}{c^2}$ is greater than 6.75.

Ex.

$$\text{Let } x^3 - 6.76x = 6.75,$$

$$\text{Let } 7x - x^3 = 7,$$

$$\text{Let } x^3 = 100x = 100;$$

these equations are in the irreducible case :

but the equations $x^3 - 6.75x = 6.75$,

$$x^3 - 6x = 6,$$

$$5x - x^3 = 5,$$

are not in the irreducible case.

RULE IV.

To find the values of the roots of the equation $bx - x^3 = c$, the root of the equation $x^3 - bx = c$ being known.

Having found the value of x , solve the quadratic equation $xt - t^2 = x^2 - b$, and the two roots of that equation will respectively be the roots of the equation $bx - x^3 = c$.

This quadratic may be converted into the following convenient expression, to give the values of the roots, the sign $+$ belonging to the greater root.

$$\frac{x}{2} \pm \frac{\sqrt{4b - 3x^2}}{2}$$

One of the roots is always greater than the other, while the equation belongs to the irreducible case; but they are equal when $\frac{b^3}{c^2} = 6.75$.

Henceforward the greater root will be called t , and the lesser root v .

Ex.

$$\text{Let } 81x - x^3 = 190.$$

The root of the equation $x^3 - 81x = 190$, is 10.

$$x^2 - 81 = 19$$

$$10t - t^2 = 19$$

$$t = 5 + \sqrt{6}$$

$$v = 5 - \sqrt{6}$$

Ex.

Let $63x - x^3 = 162$.

The root of the opposite equation is 9.

$$t \text{ and } v = \frac{9 + \sqrt{252 - 243}}{2}$$

hence $t=6, v=3$.

RULE V.

To find the value of the root of the equation $x^3 - bx = c$, when either of the roots of the equation $bx - x^3 = c$ is known.

Put d = the square of the root known,

$$e = b - d,$$

$$\text{then } x = \frac{\sqrt{d} + \sqrt{d + 4e}}{2}$$

Ex.

Let $81x - x^3 = 190$.

$$t = 5 + \sqrt{6}$$

$$d = t^2 = 31 + 10\sqrt{6}$$

$$e = 50 - 10\sqrt{6}$$

$$x = \frac{5 + \sqrt{6} + \sqrt{231 - 30\sqrt{6}}}{2} = 10$$

Ex.

Let $7x - x^3 = 6$.

$$v = 1$$

$$d = 1$$

$$e = 6$$

$$x = \frac{1 + \sqrt{25}}{2} = 3$$

RULE VI.

To find a near value of the root of the equation $x^3 - bx = c$, without assumption or trial, and at the same time to discover close limits thereof.

In the best treatise on this case that I have read, the limits are thus defined :—" x is always greater than $\sqrt{b}$, but less than $\frac{2\sqrt{b}}{\sqrt{3}}$;" and these

limits are considered as very narrow by the learned author. But much narrower limits than these may always be readily discovered. The method of obtaining them will, in the first place, be exhibited by some examples.

Ex.

Let $x^3 - 100x = 1$.

This example is favourable to the method; but many others might have been selected, wherein hundreds of correct figures would have been obtained.

Take the value of $\frac{b^3}{c^2}$, which in this case is 1000000, and its square root is 1000. Take this square root as the numerator of a fraction, and the square root +1 as the denominator. Take the denominator as the numerator of a new fraction, and this numerator +1 as the denominator.

The two fractions are $\frac{10000}{10001}$ and $\frac{10001}{10002}$.

Squaring the numerator of the second or right-hand fraction,
gives 1002001

subtracting the numerator 1001

leaves 1001000

but this last number being greater than the value of $\frac{b^3}{c^2} = 1000000$, the two fractions suit the intended purpose.

It results, that x is less than $\sqrt{\left(\frac{1001b}{1000}\right)}$

that x is greater than $\sqrt{\left(\frac{1002b}{1001}\right)}$

But in the given equation, $\frac{1001b}{1000} = 100.1$

and $\frac{1002b}{1001} = 100.0999$:

hence x is less than $\sqrt{100.1} = 10.004998$

x is greater than $\sqrt{100.0999} = 10.004993$:

consequently the figures 10.00499 are correct for a first near value of x , and the proposed narrow limits have been found.

Ex.

Let $x^3 - 3x = 1$.

This example is unfavourable to the method, because the value of $\frac{b^3}{c^2}$ is a low number.

$$\frac{b^3}{c^2} = 27$$

the square root is $5+$.

The integer only is taken, and the fractions derived, as in the former example; they are $\frac{5}{6}$ and $\frac{6}{7}$: the numerator of the last being squared, and the numerator subtracted from the square as before, there remains 30, which is greater than $\frac{b^3}{c^2} = 27$; therefore these fractions suit the intended purpose. Using the same process with the first fraction, there remains 20, which is less than $\frac{b^3}{c^2} = 27$; and it is absolutely requisite that the two remainders should be respectively below and above the value of $\frac{b^3}{c^2}$.

In this example, x is less than $\sqrt{\left(\frac{6b}{5}\right)}$, and greater than $\sqrt{\left(\frac{7b}{6}\right)}$;

therefore x is less than $\sqrt{3.6} = 1.897+$

x is greater than $\sqrt{3.5} = 1.870+$.

The two figures 1.8 must, therefore, be correct for a first near value of x ; but the proximity of the limits is not sufficient: by adding .6 to the numerator and denominator, respectively, of the first fraction, it becomes $\frac{5.6}{6.6}$.

Then $5.6^2 = 31.36$

subtract 5.6

remainder $= 25.76$, which is less than the value of $\frac{b^3}{c^2}$;

therefore x is less than $\sqrt{\left(\frac{6.6b}{5.6}\right)} = \sqrt{3.535714} + = 1.880+$, and the near value of, and the proposed narrow limits for, the root x have been found.

Ex.

Let $x^3 - 2x = 1$.

This example is as unfavourable as possible to the method, and is selected on that account.

$$\frac{b^3}{c^2} = 8$$

the square root is $2+$,

the fractions are $\frac{2}{3}$ and $\frac{3}{4}$:

but it is impossible that the former should have place in this method, because the quantity $x^3 - bx$ cannot rise so high in our case as

$\frac{x^3}{4}$, nor can ba sink to $\frac{x^3}{4}$. The fraction $\frac{2}{3}$ must, therefore, be rejected; and the two fractions $\frac{3}{4}$ and $\frac{4}{5}$ must be employed, and they have the requisite properties:

$$\text{for } 3^2 - 3 = 6$$

$$4^2 - 4 = 12$$

and the value of $\frac{b^3}{c^2} = 8$, lies between the remainders.

It may be observed, that when the first fraction is rejected, the succeeding fractions will answer the purpose; and it will not be necessary to square either of the numerators, if we are satisfied with the limits obtainable by them.

In the present example, x is less than $\sqrt{\left(\frac{4b}{3}\right)} = \sqrt{\left(\frac{8}{3}\right)} = 1.63+$, and greater than $\sqrt{\left(\frac{5b}{4}\right)} = \sqrt{\left(\frac{10}{4}\right)} = 1.5811+$.

These limits are not satisfactory, and since the first limit falls without our case, because x cannot rise to $\sqrt{\left(\frac{4b}{3}\right)}$, but may be infinitely near to it, the quantity .2 is added to the numerator and denominator, respectively, of the fraction $\frac{3}{4}$, which becomes $\frac{3.2}{4.2}$, and $3.2^2 - 3.2 = 7.94$.

The root x is therefore less than $\sqrt{\left(\frac{4.2b}{3.2}\right)} = \sqrt{\left(\frac{8.4}{3.2}\right)} = \sqrt{2.625} = 1.620$.

In the same manner, the second fraction $\frac{4}{5}$ is reduced to $\frac{3.5}{4.5}$; and $3.5^2 - 3.5 = 8.75$, which is greater than $\frac{b^3}{c^2}$. The root x is therefore greater than $\sqrt{\left(\frac{4.5b}{3.5}\right)} = \sqrt{2.5714} = 1.60+$. The figures 1.6 are correct for a first near value of x ; and the very close limits $\sqrt{\left(\frac{8.4}{3.2}\right)}$ and $\sqrt{\left(\frac{9}{3.5}\right)}$ are obtained. There are two other necessary remarks, which will be made the subjects of the next rule.

RULE VII.

To determine when the first fraction should be rejected, and to ascertain when the value of the root may be accurately found, by means of one of the fractions employed.

Should the first fraction be $\frac{2}{3}$, always reject it, and employ the fractions $\frac{3}{4}$ and $\frac{4}{5}$. The general rule will be elucidated by an example.

*Ex.*Let $x^3 - 28x = 28$.

$$\frac{b^3}{c^2} = 28$$

The fractions which first appear, are $\frac{5}{6}$ and $\frac{6}{7}$.

Attention must first be given to the second fraction, $\frac{6}{7}$; for, when the second fraction answers, we may be certain that the first is correct.

$6^2 - 6 = 30$, which is greater than the value of $\frac{b^3}{c^2}$, and the two fractions are to be used; but a nicety is about to be pointed out, not mentioned before, lest it might embarrass the subject.

*Ex.*Let $x^3 - 30x = 30$.

The fractions which first appear, are $\frac{5}{6}$ and $\frac{6}{7}$

$$6^2 - 6 = \frac{b^3}{c^2} = 30$$

The two fractions are correct, and are to be used, for the reason given in the next example.

*Ex.*Let $x^3 - 30.7x = 30.7$.

The fractions which first appear, are $\frac{5}{6}$ and $\frac{6}{7}$

$6^2 - 6 = 30$, which is less than the value of $\frac{b^3}{c^2}$; and it would seem, from what has been said, that the first fraction ought to be rejected, nevertheless these fractions are correct: for the general rule is this, that, representing the second fraction by $\frac{g}{h}$, if the value of $g^2 - g$ is nearly equal to the value of $\frac{b^3}{c^2}$, but yet a little below it, then the value of $g^2 - g + \frac{g}{h}$ must be found; and if such value is still below the value of $\frac{b^3}{c^2}$, then the first fraction must be rejected; but if such value is greater than $\frac{b^3}{c^2}$, the second fraction is correct.

In the present example, $6^2 - 6 + \frac{6}{7} = 30.8+$, which is greater than the value of $\frac{b^3}{c^2}$: consequently the value of the root x is somewhat, though very little, greater than $\sqrt{\left(\frac{7b}{6}\right)}$.

Ex.

$$\text{Let } x^3 - 30.9x = 30.9.$$

The fractions which first appear, are $\frac{5}{6}$ and $\frac{6}{7}$; but the first must be rejected, for the reason given, in the last example, respecting the second fraction; that is, because $36 - 6 + \frac{6}{7}$ is not so great as 30.9, which is the value of $\frac{b^3}{c^2}$ in this example.

The fractions to be used, are $\frac{6}{7}$ and $\frac{7}{8}$

The value of the root x is somewhat, though very little, less than $\sqrt{\left(\frac{7b}{6}\right)}$.

When the proper fractions have been found, the value of the root x is always less than the square root of the following quantity, viz. $\sqrt{\left(\frac{fb}{e}\right)}$, $\frac{e}{f}$ denoting the first fraction; and it is always greater than the square root of the following quantity, viz. $\sqrt{\left(\frac{hb}{g}\right)}$, $\frac{g}{h}$ denoting the second fraction. But this must be understood to relate to those equations only where the root cannot be accurately discovered by means of either of the fractions.

We now come to the method of finding the value of the root x accurately, by means of one or other of the fractions.

Ex.

$$\text{Let } x^3 - 90x = 100.$$

$$\frac{b^3}{c^2} = 72.9$$

the fractions are $\frac{8}{9}$ and $\frac{9}{10}$;

$$\text{but } 9^2 - 9 + \frac{9}{10} = 72.9 = \frac{b^3}{c^2}$$

Whenever this agreement occurs, the root can always be found accurately.

$$\text{In this example, } x = \sqrt{\left(\frac{10 \times 90}{9}\right)} = 10.$$

The value of the root might also be found, by multiplying the absolute term 100 by the denominator 10, and extracting the cube root; but it would be attended with more labour.

Ex.

$$\text{Let } x^3 - 72.9x = 72.9.$$

The fractions are $\frac{8}{9}$ and $\frac{9}{10}$; and $\frac{b^3}{c^2} = 72.9$

17.

$$x = \sqrt{\left(\frac{10b}{9}\right)} = \sqrt{81} = 9$$

$$\text{or, } x = \sqrt[3]{10 \times 72.9}$$

Ex.

$$\text{Let } x^3 - 17x = \sqrt{6 \frac{539}{729}}.$$

$$\frac{b^3}{c^2} = 72.9$$

The fractions are as before.

$$x = \sqrt{\left(\frac{10b}{9}\right)} = \sqrt{\left(\frac{170}{9}\right)}$$

Ex.

$$\text{Let } x^3 - 7x = 6.$$

$$\frac{b^3}{c^2} = 9.52\dot{7}$$

The fractions are $\frac{3.5}{4.5}$ and $\frac{4.5}{5.5}$

$$3.5^2 - 3.5 + \frac{3.5}{4.5} = 9.52\dot{7}$$

$$x = \sqrt{\left(\frac{4.5 \times 7}{3.5}\right)} = \sqrt{9} = 3.$$

Ex.

$$\text{Let } x^3 - 9.52\dot{7}x = 9.52\dot{7}.$$

The fractions are $\frac{3.5}{4.5}$ and $\frac{4.5}{5.5}$, and $\frac{b^3}{c^2} = 9.52\dot{7}$

$$x = \sqrt{\left(\frac{4.5 \times 9.52\dot{7}}{3.5}\right)} = \sqrt{12.25} = 3.5.$$

Ex.

$$\text{Let } x^3 - 42.875x = 42.875.$$

The fractions are $\frac{6}{7}$ and $\frac{7}{8}$.

$$7^2 - 7 + \frac{7}{8} = 42.875$$

$$x = \sqrt{\left(\frac{8b}{7}\right)} = \sqrt{\left(\frac{343}{7}\right)} = 7.$$

D

Ex.

$$\text{Let } x^3 - \left(6 + 2\sqrt{5} - 1 + \sqrt{5} + \frac{1 + \sqrt{5}}{2 + \sqrt{5}}\right)x = 5 + \sqrt{5} + \frac{1 + \sqrt{5}}{2 + \sqrt{5}}.$$

$$x = 1 + \sqrt{5}$$

for the equation is equivalent to the equation $x^3 - 8x = 8$.

Ex.

$$\text{Let } x^3 - \left(.25 - .5 + \frac{.5}{1.5}\right)x = .08\bar{3}.$$

$$x = .5.$$

This example is not in the irreducible case.

The only integral numbers that admit of such development, are 1, 2, 3, 4, 5, 6, and 8; that is, if we make $x^2 - x + \frac{x}{x+1}$, equal to any other integral number, the value of x cannot be found; and it is this inability which forms one of the chief obstacles to the reduction of our case.

It is obvious, that an equation of the form $x^3 - bx = b$, is equivalent to the equation $x^3 - \frac{x^4}{x+1} = \frac{x^3}{x+1}$;

$$\text{for } \frac{x^5 + 2x^4 + x^3}{x^2 + 2x + 1} = x^3.$$

The principle on which the foregoing rule is founded, may be thus explained. Let there be given for solution, the equation $x^3 - bx = c$, and let this be converted into another, of the form $y^3 - \frac{b^3}{c^2}y = \frac{b^3}{c^2}$; the

value of y will be $\sqrt{\left(\frac{x^2 \times \frac{b^3}{c^2}}{-\frac{b}{b}}\right)} = \frac{bx}{c}$; and the value of x will be

$\sqrt{\left(\frac{y+1 \times b}{y}\right)}$, or $x = \frac{cy}{b}$. Substituting the quantity $n^2 - n + \frac{n}{n+1}$ for $\frac{b^3}{c^2}$, the second equation becomes

$$y^3 - \left(n^2 - n + \frac{n}{n+1}\right)y = n^2 - n + \frac{n}{n+1} :$$

the value of y , the root of this equation, is n ; but the square root of the whole quantity, $n^2 - n + \frac{n}{n+1}$, cannot be so great as n , because the value of the root y must be greater than the square root of the coefficient. If we take z as the square root of the coefficient, and as the numerator of a fraction, and add unity to z as the denominator, we get the fraction $\frac{z}{z+1}$; but $z^2 - z + \frac{z}{z+1}$, is less in value than

$n^2 - n + \frac{n}{n+1}$; and consequently y is less than $\sqrt{\left(\frac{z+1 \times n^2 - n + \frac{n}{n+1}}{z}\right)}$

for $y=n$ is exactly equal to $\sqrt{\left(\frac{n+1 \times n^2 - n + \frac{n}{n+1}}{n}\right)}$, the quantity

$\frac{n+1 \times n^2 - n + \frac{n}{n+1}}{n}$ being equal to n^3 .

Again, if we take another quantity, w greater than z , and so that the value of $w^2 - w + \frac{w}{w+1}$, shall be greater than the value of

$n^2 - n + \frac{n}{n+1}$; then y will be greater than $\sqrt{\left(\frac{w+1 \times n^2 - n + \frac{n}{n+1}}{w}\right)}$,

for this last quantity will be less than n .

In order to discover the limits of the root x , by means of those obtained for y , if we put $d = \frac{b^3}{c^2}$, then $x = \sqrt{\left(\frac{y^2 \times b}{d^2}\right)} = \sqrt{\left(\frac{y+1 \times b}{y}\right)}$, for

$\frac{d}{y^2} = \frac{y}{y+1}$; and it results, that the limits for the value of x are obtainable

by the application of the same quantities that are used to discover those for y ; because the quantity $\frac{b^3}{c^2} y$ has the same ratio to y^3 , as bx has to x^3 . A more detailed investigation would lead us into a field too extensive for the purposes of this work.

RULE VIII.

To discover the roots of several equations, the value of the root x of one equation being known; the value of $\frac{b^3}{c^2}$ being the same in all the equations.

Multiply the coefficients of the given equations by the square of the root of the equation, the value of whose root is known, and divide the products by the coefficient of the same equation. The square roots of the quotients will be the roots required.

Ex.

Let the equation, the root of which is known, be

$$x^3 - 90x = 100$$

$$x = 10$$

$$\frac{b^3}{c^2} = 72.9,$$

D 2

Let the following equations be solved by the rule.

$$x^3 - x = \frac{100}{90\sqrt{90}}$$

$$x^3 - 3x = \frac{100\sqrt{27}}{90\sqrt{90}}$$

$$x^3 - 72.9x = 72.9$$

The value of $\frac{b^3}{c^2}$, is 72.9 in each equation.

The value of x in the first equation, is $\sqrt{\left(\frac{10}{9}\right)}$

of x in the second equation, is $\sqrt{\left(\frac{30}{9}\right)}$

of x in the third equation, is $\sqrt{\left(\frac{72.9}{9}\right)} = 9$.

Hence it appears, that the ratio of bx to x^3 , depends on the magnitude of $\frac{b^3}{c^2}$; for, as the magnitude of $\frac{b^3}{c^2}$ increases, the ratio of bx to x^3 increases accordingly.

This rule is applicable to all those equations of the form $x^3 - bx = c$, which can be solved by Tartaglia's rule, and also to equations of the form $x^3 + bx = c$.

RULE IX.

To generate an equation in the irreducible case with any coefficient, and with any root that is consistent with the magnitude of the coefficient.

Divide the coefficient into any two parts, and make the absolute term equal to the product derived from the multiplication of the square root of one of the parts by the other part. Such equation will be in the irreducible case, except the square root of one-third part of the coefficient is taken; for then the roots t and v will be equal, and the equation will be solvable by Tartaglia's rule.

Ex.

Let 17 be the coefficient selected.

Let it be divided into the parts 4 and 13, and let 4 be the part, the square root of which is to be multiplied by the other part. The equation becomes $x^3 - 17x = 26$.

$$x = \frac{\sqrt{4} + \sqrt{4 + 4 \times 13}}{2} = 1 + \sqrt{14}$$

$$t = \sqrt{14} - 1, v = 2.$$

Let the coefficient be divided as before, and let 13 be the part, the square root of which is to be multiplied by the other part. The equation becomes $x^3 - 17x = \sqrt{208}$.

$$x = \frac{\sqrt{13} + \sqrt{13 + 4 \times 4}}{2} = \sqrt{3.25} + \sqrt{7.25}$$

$$t = \sqrt{13}, v = \sqrt{7.25} - \sqrt{3.25};$$

and, in general, if d is substituted for the part, the root of which is to be taken, and e for the other part, then

$$x = \frac{\sqrt{d} + \sqrt{d + 4e}}{2}$$

This property, independently of the great use it is of in the generation of equations with the coefficient 3, frequently enables us to find the roots of equations very difficult to be solved otherwise.

Ex.

Required the roots of the equations $x^3 - 22x = \sqrt{1575}$.

$$\sqrt{1575} = 15\sqrt{7}$$

$$15 + 7 = 22$$

$$\text{therefore } x = \frac{\sqrt{7} + \sqrt{7 + 4 \times 15}}{2} = \sqrt{1.75} + \sqrt{16.75}$$

$$t = \sqrt{16.75} - \sqrt{1.75}, v = \sqrt{7}.$$

The reader will observe, that the roots t and v are given with positive signs, because they are the affirmative roots of the opposite equation $bx - x^3 = c$; and this must be continually understood as we proceed.

Having found the value of x , the values of t and v are apparent, for $\sqrt{d}$ is equal either to t or to v ; and since t is always greater than $\frac{x}{2}$, no mistake can occur.

If $t = \sqrt{d}$, then $x - t = v$; if $v = \sqrt{d}$, then $x - v = t$.

Consequently, when x is discoverable, its value may always be found under two forms.

Ex.

$$\text{Let } x^3 - 7x = 6$$

$$d = 1, e = 6$$

$$d = 4, e = 3$$

$$x = \frac{\sqrt{1} + \sqrt{1+24}}{2} = 3$$

$$x = \frac{\sqrt{4} + \sqrt{4+12}}{2} = 3.$$

There is, however, no rule whereby we can find e universally; such a rule would perfect the solution of the case.

It is necessary to remark here, that x and v are at their greatest magnitudes, and t at its least magnitude, when the absolute term is infinitely near in value to $\sqrt{\left(\frac{4b^3}{27}\right)} = \sqrt{\left(\frac{b}{3}\right)} \times \frac{2b}{3}$.

There is another very easy method of making an equation in the irreducible case, of the form $x^3 - bx = c$. Take any two numbers for t and v , t being the greater number, and their sum for x :

then $\frac{x^2 + t^2 + v^2}{2}$ will be the coefficient,

and $x t v$ will be the absolute term.

Ex.

Let $t = 6$	$t^2 = 36$	
$v = 4$	$v^2 = 16$	$x t v = 240$
$x = 10$	$x^2 = 100$	

$$2) 152 (76$$

the equation becomes

$$x^3 - 76x = 240$$

$$\text{or, } 76x - x^3 = 240.$$

The rule for the absolute term is well known; but the expression for the coefficient is, perhaps, new.

RULE X.

To decompose any number or quantity into three squares, in an infinite number of ways, and in such manner that one of the roots shall be equal to the sum of the other two roots.

Take any number or quantity as the coefficient of the equation $x^3 - bx = c$. Equations having such number or quantity as a coefficient, may be infinite in number, for the absolute term may be infinitely varied; but in every equation of the form $x^3 - bx = c$, belonging to our case, as well as in that where the value of $\frac{b^3}{c^2}$ is exactly 6.75, $x^2 + t^2 + v^2 = 2b$.

These squares cannot be discovered universally, yet they may be discovered in an infinite number of ways. The means of effecting it may be elucidated by an example.

Ex.

Let it be required to find 3 squares equal to the number 20.

Take 10 for the coefficient of the equation $x^3 - bx = c$. Take any number greater than $\sqrt{b} = \sqrt{10}$, but not greater than $\sqrt{\left(\frac{4b}{3}\right)} = \sqrt{\left(\frac{40}{3}\right)}$; for the value of the root x .

$$\begin{array}{lcl}
 & \text{Let such value be } 3.5, & \\
 & \text{then } x^3 - 10x = 7.875 & \\
 x = 3.5 & t = 1.75 + \sqrt{.8125} & \\
 & \tau = 1.75 - \sqrt{.8125} & \\
 x^2 = 12.25 & & \\
 t^2 = 3.875 + 3.5\sqrt{.8125} & & \\
 \tau^2 = 3.875 - 3.5\sqrt{.8125} & & \\
 \hline
 & 20 &
 \end{array}$$

If any other numbers within the prescribed limits are used for the value of x , the squares may be discovered by means of the foregoing rule.

RULE XI.

An equation of the form $x^3 + dx = c$, being given, to find an equation of the form $bx - x^3 = c$, having the same root and absolute term.

Find the root of the former equation by the known rule.

Then $2x^2 + d = b$.

Ex.

$$\text{Let } x^3 + 4x = 240 \quad x = 6$$

$$\text{Let } x^3 + 44x = 240 \quad x = 4$$

$$72 + 4 = 76$$

$$32 + 44 = 76,$$

the resulting equation is $76x - x^3 = 240$,

$t = 6$, $\tau = 4$, and the absolute term is unaltered.

From this connection between the two forms, innumerable and otherwise most difficult equations of the form $bx - x^3 = c$, may be solved; and by the converse of the rule, very complicated expressions may be brought to a simple form.

Ex.

$$\sqrt[3]{\frac{3}{2} + \sqrt{\frac{9}{4} + \frac{\sqrt{45+1}^3}{27}}} - \sqrt[3]{\sqrt{\frac{9}{4} + \frac{\sqrt{45+1}^3}{27}} - \frac{3}{2}} = 1.5 - \sqrt{1.25}:$$

for, let $8x - x^3 = 3$

$$x = 1.5 - \sqrt{1.25}.$$

To convert this equation into one of the form $x^3 + dx = c$, put $b - 2v^2 = d$, the resulting equation is

$$x^3 + \sqrt{45+1} \times x = 3,$$

the root of which is also $1.5 - \sqrt{1.25}$, and equal to the value of the preceding complicated expression.

The equation $bx - x^3 = c$ has always a corresponding equation, $x^3 + dx = c$, with the lesser root v , and same absolute term: but it is not always so with t , or the greater root; for the corresponding equation having the same root and absolute term, may be of the forms $x^3 + dx = c$, $x^3 = c$, or $x^3 - dx = c$; but always reducible in this last case.

Ex.

$$\text{Let } 103x - x^3 = 198$$

$$x = 2, t = 9$$

$$b - 2v^2 = 95$$

$$b - 2t^2 = -59,$$

the corresponding equations are

$$x^3 + 95x = 198, x = 2$$

$$x^3 - 59x = 198, x = 9.$$

$$\text{Again, let } 8x - x^3 = 8,$$

the corresponding equation is

$$x^3 = 8:$$

hence, it results, that v , the lesser root of the equation $bx - x^3 = c$

$$= \sqrt[3]{\frac{c}{2} + \sqrt{\frac{6.75c^2 + b^3 - 6b^2v^2 + 12bv^4 - 8t^6}{27}}} - \sqrt[3]{\sqrt{\frac{6.75c^2 + b^3 - 6b^2v^2 + 12bv^4 - 8v^6}{27}} - \frac{c}{2}}$$

RULE XII.

To generate equations of the forms $x^3 - bx = c$, $bx - x^3 = c$, with any coefficients, so that the roots t and v of equations of the latter form may have a constant ratio to each other, and to the root x of equations of the first form, and so that the values of $\frac{b^3}{c^2}$ may be the same in all the generated equations.

Take any whole number, surd, or fraction, for the coefficient of an equation; divide it into any two parts, d and e , by Rule IX. Make the absolute term equal to $\sqrt{de}$; then whatever ratio t and v have to each other, and respectively to x , in such equation, the same ratios will exist in all other equations of the same form where the coefficients shall be divided in like manner, and the absolute terms made accordingly; and the values of $\frac{b^3}{c^2}$ will be the same in all the equations.

Ex.

Let the coefficient be 28,
divide it into the parts 4 and 24,
the equation becomes $x^3 - 28x = 48$.

$$\frac{28^3}{48^2} = 9.527$$

$$x = \frac{\sqrt{4} + \sqrt{4 + 96}}{2} = 6$$

$$t = 4, v = 2.$$

Again, let the coefficient be 63.

Take $\frac{1}{7}$ thereof, as in the last example, for d ,

the parts are 9 and 54,
the equation becomes $x^3 - 63x = 162$.

$$x = \frac{\sqrt{9} + \sqrt{9 + 216}}{2} = 9$$

$$t = 6, v = 3.$$

The ratios are the same in each equation, and the values of $\frac{b^3}{c^2}$ are alike.

If the coefficient is divided into two equal parts, the value of $\frac{b^3}{c^2}$ will be 8.

RULE XIII.

To discover the roots of the equations

$$w^3 - 3\sqrt{b}w^2 + 2bw = c,$$

$$z^3 + 3\sqrt{b}z^2 + 2bz = c,$$

the roots of the equations $x^3 - bx = c$, $bx - x^3 = c$, being known.

$$\text{Put } w = \sqrt{b} - v, \sqrt{b} + x, \sqrt{b} - t,$$

$$z = x - \sqrt{b}.$$

E

The former equation has always three real and affirmative roots, when thus related to the irreducible case. The latter equation has but one such root.

Ex.

Let it be required to find the three roots of the equation

$$w^3 - 3\sqrt{9}w^2 + 18w = 8,$$

the roots of the equations $x^3 - 9x = 8$, $9x - x^3 = 8$, being known.

$$x = .5 + \sqrt{8.25}, t = \sqrt{8.25} - .5, v = 1 :$$

$$\text{then } w = 3.5 + \sqrt{8.25}, 3.5 - \sqrt{8.25}, 2.$$

It has been shewn in Rule II. that the quantity $x^3 - bx$ is equal to the product of three numbers in arithmetical progression, their common difference being $\sqrt{b}$. The three numbers in this case are,

$$.5 + \sqrt{8.25} - 3, .5 + \sqrt{8.25}, 3.5 + \sqrt{8.25}.$$

The last or greatest of these numbers, that is, $x + \sqrt{b}$, is always one of the three roots of the equation $w^3 - 3\sqrt{b}w^2 + 2bw = c$.

Ex.

Let it be required to find the root of the equation

$$z^3 + 3\sqrt{9}z^2 + 18z = 8,$$

the root of the equation $x^3 - 9x = 8$ being known, and which is $.5 + \sqrt{8.25}$:

then $z = x - \sqrt{b} = \sqrt{8.25} - 2.5$; and consequently z is equal to the first or least number in the arithmetical progression before mentioned. Of these four roots, there are two which more peculiarly appertain to our case, and, by their complicated functions, stand in the way of the reduction of our problem.

The two roots are $x - \sqrt{b}$ and $\sqrt{b} - t$, or the least value of π . The sum of these roots is equal to v , or the least root of the equation $bx - x^3 = c$.

RULE XIV.

To discover the roots of the equation $be^2 - e^3 = c^2$, the roots of the equation $bx - x^3 = c$ being known.

$$\text{Put } e = b - t^2$$

$$e = b - v^2$$

The equation belongs to the irreducible case, except when $\frac{b^3}{c^2}$ is equal to 6.75; and it cannot have any real and affirmative roots, unless $\frac{b^3}{c^2}$ is equal to or greater than 6.75.

Ex.

Let the roots of the equation $28e^2 - e^3 = 2304 = 48^2$ be required, and let the roots of the equation $28x - x^3 = 48$ be known.

$$t=4, v=2:$$

$$\text{then } e=28-16=12, e=28-4=24.$$

When the roots t and v are equal, then the roots e, e , are equal.

It is by no means improper to say, that equations have even roots; for, in constructions of cubic equations, the roots are distinct even when they are equal.

RULE XV.

To ascertain, very frequently, by inspection, that the roots of the equations $x^3 - bx = c$, $bx - x^3 = c$, may be accurately found.

Compare the magnitudes of b and c . If you find

$$c = b - 1; \text{ then } x = .5 + \sqrt{b - .75}$$

$$c = \sqrt{b-1}; \text{ then } x = \frac{\sqrt{b-1} + \sqrt{b+3}}{2}$$

$$c = \sqrt{b+1}; \text{ then } \cancel{t} = \frac{\sqrt{b+1} + \sqrt{b-3}}{2}$$

and, in general, if it can be perceived that

$$c = 2b - 8$$

$$c = 3b - 27$$

$$c = 4b - 64$$

$$c = nb - n^3$$

the roots may be found by Rule IX.; that is, d and e may be discovered.

Ex.

$$\text{Let } x^3 - 71x = 186$$

$$c = 3b - 3^3$$

$$d=9, e=62$$

$$x = \frac{\sqrt{9} + \sqrt{9 + 4 \times 62}}{2} = 1.5 + \sqrt{64.25}$$

$$t = \sqrt{64.25} - 1.5, v=3.$$

Another method of frequently discovering the roots, may be added.

In the equation $x^3 - bx = c$,

$$x^2 + 1 = \sqrt{(b+2) \times x^2 + cx + 1}$$

The root of a proposed equation may not be discoverable by means of this formula; yet by making use of it in respect of another equation having the value of $\frac{b^3}{c^2}$, the same as in the proposed equation, the root may sometimes be found.

Let the equation to be solved be

$$x^3 - \sqrt[3]{9.527}x = 1.$$

Here the application of the formula will not succeed; therefore make an equation having the absolute term equal to $2x$, and likewise having the value of $\frac{b^3}{c^2}$ equal to 9.527 . The equation agreeing with these conditions, is $y^3 - 7y = 6$.

$$y^2 + 1 = \sqrt{9y^2 + 6y + 1} = 3y + 1 : \\ \text{hence } y = 3.$$

By Rule VIII. the root x of the proposed equation is $\sqrt{\left(\frac{9 \times \sqrt[3]{9.527}}{7}\right)}$

This method will succeed in innumerable cases, all different from the present; but it would lead us from our subject to dilate on it.

RULE XVI.

To find the cube root of a cube number, when computed tables at hand are not sufficiently extensive.

Divide the number by 216, and if there is a remainder, divide it by 6, and reserve the remainder of this last division. Find the cube number next below the first quotient in the tables. Six times the cube root of that cube number, together with the reserved remainder, will be the required cube root.

Ex.

Let the cube root of 7610498063 be required.

$$\frac{\text{Cube}}{216} = 35233787, \text{ with remainder } 71$$

$$\text{remainder of } \frac{71}{6} = 5.$$

The cube number next below 35233787, is 34965783, the cube root of which is 327.

$$6 \times 327 + 5 = 1967, \text{ the required cube root.}$$

There are many other ways of effecting the same object, but this is very expeditious, and, perhaps, more so than any other; besides, the same method with other divisors applies to all pure powers of numbers.

RULE XVII.

To find expressions, similar to that promulgated by Cardan, for a first near value of the root of the equation $x^3 - bx = c$, by means of the fractions made use of in Rule VII.

Represent the greater of the fractions by $\frac{g}{h}$: then the first near value of x will be

$$.5\sqrt{h} \times \sqrt[3]{\frac{c}{\sqrt{h}} + \sqrt{\frac{c^2}{h} - \frac{b^3}{g^3}}} + .5\sqrt{h} \times \sqrt[3]{\frac{c}{\sqrt{h}} - \sqrt{\frac{c^2}{h} - \frac{b^3}{g^3}}}$$

If $g^2 - g + \frac{g}{h}$ is equal to $\frac{b^3}{c^2}$, the expression will be correct for the value of the root x ; if it is only a little more than the value of $\frac{b^3}{c^2}$, the approximation will be considerable; but if it is much greater than such value, the approximation will diminish accordingly: but this may be avoided, because $g^2 - g + \frac{g}{h}$ may always be taken nearly equal in value to $\frac{b^3}{c^2}$.

In all cases, except when the expression is correct, the value found will exceed the real value of x .

Ex.

$$\text{Let } x^3 - 90x = 100.$$

The fractions are $\frac{8}{9}$ and $\frac{9}{10}$

$$x = .5\sqrt{10} \times \sqrt[3]{\frac{100}{\sqrt{10}} + \sqrt{\frac{10000}{10} - \frac{729000}{729}}} + .5\sqrt{10} \times \sqrt[3]{\frac{100}{\sqrt{10}} - \sqrt{\frac{10000}{10} - \frac{729000}{729}}}$$

Hence $x = 10$, and the expression is correct.

Ex.

$$\text{Let } x^3 - 210.93749x = 210.93747.$$

The fractions are $\frac{14}{15} + \frac{15}{16}$

$$x = .5\sqrt{16} \times \sqrt[3]{\frac{c}{4} + \sqrt{\frac{c^2}{16} - \frac{b^3}{3375}}} + .5\sqrt{16} \times \sqrt[3]{\frac{c}{4} - \sqrt{\frac{c^2}{16} - \frac{b^3}{3375}}}$$

Whoever will take the trouble to discover the value of the root x by means of this expression, will obtain many correct figures: but the use of such expressions is not recommended; for the value of

the root may be more commodiously and accurately obtained by means of the fractions, agreeably to the means prescribed in Rule VII.

The equation $x^3 - bx = c$ cannot be solved generally by expressions similar to these; the principle from which they are deduced, is as adverse as possible to the reduction of our case.

It may be observed, however, that the near value of x obtained by the expression given in this rule, will never appear under an imaginary form, because the quadratic binomial will always be possible.

RULE XVIII.

To find a first near value of the root of the equation $x^3 - bx = b$.

The method of approximation about to be introduced, depends on the finding a first near value of the root of this equation: two means of effecting it, without trial or conjecture, will be given, and the reader will be enabled to select the most appropriate, when he has gone through the examples.

If b denotes a number of considerable magnitude, solve the equation $x^3 - bx = b - 1$, the root of which can always be found, and you will have many correct figures for the root of the equation $x^3 - bx = b$.

Ex.

Let $y^3 - 300000y = 300000$.

Find the root of the equation $z^3 - 300000z = 299999$.

$z = .5 + \sqrt{299999.25}$, by Rule XV.;

or, $z = 548.2218 +$.

All these figures are correct for a first near value of the root y , and the same number of correct figures will always be obtained when b denotes a number as great as in the present example; but to be certain of the correctness of these figures, let a near value of the root y be found by means of the fractions.

The fractions are $\frac{548.2}{549.2}$ and $\frac{548.3}{549.3}$

for $548.2^2 - 548.2 = 299975.24$

$548.3^2 - 548.3 = 300084.89$

y is less than $\sqrt{\left(\frac{549.2 \times 300000}{548.2}\right)} = \sqrt{300547.24553082} + = 548.22180 +$

y is greater than $\sqrt{\left(\frac{549.3 \times 300000}{548.3}\right)} = \sqrt{300547.14572312} + = 548.22180 +$

therefore these seven figures and the cypher are assuredly correct for an approximation to the value of y , and they agree with those ob-

tained for the value of z to seven places of figures; but those obtained for the value of z were found with much less labour.

If the value of b is greater than 30, we shall get two figures of the root by solving the equation $x^3 - bx = b - 1$; and as the numerical value of b increases, so will a greater approximation take place.

But a certain and most speedy method of finding a first near value of the root of the equation $x^3 - bx = b$, is as follows:

To the root of b add .4 for such near value; and that such near value is correct, may be thus shewn:

Let it be supposed that the value of the root x is $\sqrt[3]{b} + .5$:

$$\text{then } x^3 = b\sqrt[3]{b} + 1.5b + .75\sqrt[3]{b} + .125$$

$$\text{subtract } bx = b\sqrt[3]{b} + .5b$$

leaves

$$b + .75\sqrt[3]{b} + .125$$

but this remainder is greater than b , and consequently the value of x cannot be so great as $\sqrt[3]{b} + .5$.

And that the value of x cannot be so little as $\sqrt[3]{b} + .4$, while the equation belongs to our case, may be thus shewn:

The equation which limits this case, is $x^3 - 6.75x = 6.75$, where $x = 3$;

$$\text{but } \sqrt[3]{6.75} = 2.59807 +$$

$$\text{adding } .4$$

$$\text{gives } 2.99807 +$$

which is not sufficient for the value of the root x ; and since the root x , in the equation $x^3 - bx = b$, must always be greater than 3, it follows, from the nature of cube numbers, that the value of such root must be greater than $\sqrt[3]{b} + .4$; but it is not a consequence that we should universally derive two correct figures of the root by this means: the knowledge, however, of these limits is sufficient to enable us to continue the approximation with absolute certainty.

There is another means of finding a near value of the root x , depending on trial or conjecture; but the conjecture, or trial, lies within very narrow limits.

In the equation $bx - x^3 = b$, the lesser root v cannot be so little as 1, nor can it be so great as 1.5, while the equation is in the irreducible case; for it is only equal to 1.5 in the equation $6.75x - x^3 = 6.75$, which is the limiting equation, where the roots t and v are equal to each other.

The equation $\overline{6.75 - z} \times x - x^3 = 6.75 - z$ is impossible, z being an infinitely small number; but the equation $\overline{6.75 + z} \times x - x^3 = 6.75 + z$ is

the commencement of the irreducible case, because the absolute term is less than $\frac{x^3}{4}$. The roots t and v are now of unequal magnitudes; t is greater than 1.5, and v is less than 1.5, continually approaching unity, but yet not capable of arriving at it. The root t may be of any magnitude whatever, because there is no limit to the possible magnitude of b .

Since the root v cannot be so little as 1, nor so great as 1.5, while the equation is in the irreducible case, we may avail ourselves of the knowledge of these limits, and by the aid of Rule IX. find a first near value of the root x , which may be exemplified thus:

Ex.

Let $x^3 - 80x = 80$.

Divide the coefficient into two parts, whereof one part shall be greater than unity, but less than 1.5, and in such manner, that the product of the square root of the lesser part and the greater part, shall be nearly equal to the absolute term 80.

Take 1.014 as the square root:

then $d = 1.028196$

$e = 78.971804$

80

$\sqrt{de} = 80.077409256$.

Make an *auxiliary* equation $z^3 - 80z = 80.077409256$;

then $z = \frac{1.014 + \sqrt{316.901216}}{2} = 9.40785 +$.

The four figures 9.407 are correct for a first near value of the root of the equation $x^3 - 80x = 80$; but the next figure 8 is in excess, the nearer value of the root x being 9.4077.

As this method of obtaining a near value of the root x depends on trial or conjecture, it must not be considered as forming an essential part of the method introduced in this work.

RULE XIX.

To find the value of the root of the equation $x^3 - bx = c$, when the root is a whole number.

This may be always easily effected. It has been shewn in Rule II. that the quantity $x^3 - bx$, and consequently its equal c , is equal to the product of three numbers in arithmetical progression, their common

difference being $\sqrt{b}$, and the middle number being equal to x ; and we are enabled by Rule III. to determine whether such progression is connected with the irreducible case or not.

Therefore, when an equation of this form is given for solution, and it is required to ascertain whether x is a whole number or not; then, if b should not be a natural square number, take the square root of the next rational square, either above or below the value of b , and set it down as the middle number of the progression: find the product of the three terms, and if such product is near in value to the absolute term, add to it the product of b and such middle number, if the higher square root has been taken; or subtract the product of b and such middle number, if the lower square root has been taken: should the result be equal to the absolute term, the middle number will be the value of x . The following examples will elucidate the rule.

Ex.

Let $x^3 - 80x = 200$.

Taking the square root of 81, the series becomes

1, 10, 19:

the product of the three terms is 190

add the middle term . 10

the sum is 200

and $x = 10$.

Let the same equation be tried with the lower square 64.

The series becomes 1, 9, 17:

the product of the three terms is 153; but as this product is considerably beneath the value of the absolute term, each term in the series must be increased by unity, whereby a new series results, namely, 2, 10, 18, the product of which is 360, which is greater than the absolute term 200.

These three last terms are connected with the equation $x^3 - 64x = 360$, the root of which is 10; and if from this product 360

we subtract $16x = 160$

there remains 200

and consequently 10 is the root of the equation $x^3 - 80x = 200$.

Ex.

Let $x^3 - 990x = 5644$.

The coefficient is not a square number; taking the nearest rational square root, $31 = \sqrt{961}$, as the common difference of the three terms, the serieses are,

1, 32, 63

2, 33, 64

3, 34, 65

4, 35, 66

and selecting that series, the product of which is nearest in value to 5644, namely, $3 \times 34 \times 65 = 6630$

and subtracting $29 \times 34 = 986$

there remains $\underline{5644}$

hence $x = 34$.

Ex.

Let $x^3 - 134689x = 435297600$.

The coefficient is a square number, and when that is the case, unity must be added to the square root, for the middle term of the progression.

The first series is 1, 368, 735 :

but as the product of the three terms is much beneath the value of the absolute term, it is evident that serieses containing much higher numbers must be made.

Let such serieses be 100, 467, 834

200, 567, 934

300, 667, 1034

400, 767, 1134

500, 867, 1234

If the root is a whole number, it must lie between the numbers 767 and 867, because the product of the series, whereof the former is the middle term, is less than the absolute term; and the product of the series, whereof the latter number is the middle term, is greater than the absolute term.

Confining the investigation to these limits, the series 450, 817, 1184 is obtained, the product of which is equal to the absolute term, and consequently $x = 817$.

The equation does not belong to the irreducible case.

Ex.

Let $x^3 - 60x = 6800$.

The serieses in succession are, 1, 9, 17

5, 13, 21

10, 18, 26

12, 20, 28

the product of the last is 6720 :

but the series is connected with the equation $x^3 - 64x = 6720$

adding $4x = 4 \times 20 = 80$

the sum is 6800

hence $x = 20$.

The equation is not in the irreducible case.

It is obvious, that the rule extends to equations of the form $x^3 - bx = c$, where the coefficients and absolute terms are surds : one example of this kind will be given.

Er.

Let $x^3 - (11 + \sqrt{30})x = \sqrt{180} + \sqrt{150}$.

The series becomes

$\sqrt{5} + \sqrt{6} - \sqrt{(11 + \sqrt{30})}$, $\sqrt{5} + \sqrt{6}$, $\sqrt{5} + \sqrt{6} + \sqrt{(11 + \sqrt{30})}$;

the product of the three terms is equal to the absolute term :

hence $x = \sqrt{5} + \sqrt{6}$.

EXEMPLIFICATION OF THE METHOD.

IT has been shewn in Rule VII. that an equation of the form $x^3 - bx = c$, may be converted into an equation of the form $y^3 - \frac{b^3}{c^2}y = \frac{b^3}{c^2}$, where $x = \frac{cy}{b}$, or $\sqrt{\left(\frac{y+1}{y} \times b\right)}$. It has also been shewn, that in an equation of the form $y^3 - by = b$, such equation being in the irreducible case, the root y cannot be so little as 3; consequently the quantity $\frac{1}{y+1}$ cannot be so little as $\frac{1}{4}$.

If, then, an equation of the form $x^3 - bx = c$, is given for solution, and it is transformed into an equation of the form $y^3 - \frac{b^3}{c^2}y = \frac{b^3}{c^2}$, or substituting d for $\frac{b^3}{c^2}$, into an equation of the form $y^3 - dy = d$, the following equation results; namely,

$$\sqrt{\left(d - .75 + \frac{1}{y+1}\right)} + .5 = y.$$

The method is founded on this equation.

When d is represented by a low number, then the equation $y^3 - dy = d$ is solved with the greatest difficulty, because the quantity $\frac{1}{y+1}$ is immediately brought into action: whereas, when d is a high number, y is a high number also, and the quantity $\frac{1}{y+1}$ may not begin to act until several correct figures for the root y have been found.

Having obtained a first near value of y to two places of figures, by means of the preceding rules, and which it has been shewn can be effected without trial or conjecture, from the quantity d take .75, and to the remainder add the quantity $\frac{1}{y+1}$ to four decimal places, y being equal to the first discovered value. Take the square root of the resulting quantity, namely, $d - .75 + \frac{1}{y+1}$, as far as four figures, to which add .5; and two new correct figures for the value of y will

almost always be obtained thereby. But if the last figure should be incorrect, and there can be no incorrect figure but the last in any state of the approximation, it will be in excess, and the next operation will correct it; so that the erroneous figure will by no means impede the approximation.

Repeat the operation with the four figures obtained for the value of y , in order to obtain two other figures, and proceed in a similar manner until ten or eleven figures are obtained. To this extent the approximation may be absolutely confided in; observing, however, that if a doubt exists respecting the accuracy of the last figure, it may be proved correct, or rendered so, by repeating the operation with the figures obtained. I do not know of any obstacle, nor do I think that any exists, to prevent the approximation from being carried much further with absolute certainty.

There is no equation of the form $x^3 - bx = c$, that cannot be solved to the foregoing extent in less than half an hour. Eight figures of the root may be found in twenty minutes, and six figures in about twelve minutes. This trial I have frequently made with success before competent judges; and I venture to say, that any boy, quick in multiplication and division, and in the extraction of the square root, will be able to effect the same.

The examples about to be introduced, will exhibit the method in its application to every necessary variety of equations.

In the Appendix to "The Principles of Algebra," a work of great merit by Mr. Frend, there are several equations of the form $x^3 - bx = c$, solved by the venerable and learned Baron Maseres, according to Mr. Raphson's method of approximation. These will be solved in the first place, and I request the operations may be compared. The reader will observe, that when an equation of the form $bx - x^3 = c$, is proposed to him for solution, he may solve the equation $x^3 - bx = c$ by this method, and find the two roots of the former equation by Rule IV., which will take him only a few minutes to accomplish.

Ex. I.

Let $x^3 - 50x = 120$.

$$\frac{b^3}{c^2} = 8.680\dot{5}$$

the equation to be first solved, and which henceforward will be called the cotemporary equation, is $y^3 - 8.680\dot{5}y = 8.680\dot{5}$

$$\sqrt{8.680\dot{5}} = 2.94+$$

by Rule XVIII. adding .4

gives 3.3 as a first value of y ; and these two figures are to be applied as directed, in the following manner:

8.680555

subtracting .75

leaves 7.930555

$$\text{adding } .232558 = \frac{1}{4.3}$$

gives 8.163113, the square root of which is 2.857

adding .5

Again,

gives 3.357 as a 2d value of y .

7.9305555555

$$.2295157218 = \frac{1}{4.357}$$

8.1600712773, the root is 2.85658

adding .5

gives 3.35658 as a third value of y .

Again,

7.93055555555555

$$.22953784849583 = \frac{1}{4.35658}$$

8.16009340405138, the root is 2.8565877

adding .5

gives 3.3565877 as a fourth value of y .

Presuming these figures are correct, I take

$$x = \sqrt{\left(\frac{y+1 \times 50}{y}\right)} = \sqrt{64.896080325861886462} = 8.055810345, \text{ which}$$

figures are all correct. More figures have been taken at this step than strictly warrantable; but in this first example, it was desirable to shew the power of the convergency.

If x had been taken equal to $\frac{cy}{b}$, only seven correct figures would have been obtained; and the reader will please to remark, that he will almost always obtain more correct figures for the root x , by adhering to the manner in which the present example has been solved, than by taking x equal to $\frac{cy}{b}$.

Ex. II.

Let $x^3 - 5x = 4$.

$$\frac{b^3}{c^2} = 7.8125$$

the cotemporary equation is $y^3 - 7.8125y = 7.8125$

$$\sqrt{7.81} + = 2.79 +$$

adding .4

$$3.19$$

Here, as the figure next following 3.1 is a high decimal, I find the value of $3.2^2 - 3.2 + \frac{3.2}{4.2}$; and since this value is less than 7.8125, 3.2 is less than the value of y : and these two figures are to be employed, as the figures 3.3 were employed in the last example.

7.8125

.75

7.0625

$$.238095 = \frac{1}{4.2}$$

7.300595, the root is 2.702

adding .5

gives 3.202 as a second value of y .

Again,

7.0625

$$.2379819133 = \frac{1}{4.202}$$

7.3004819133, the root is 2.70194

adding .5

gives 3.20194 as a third value of y .

Again,

7.0625

$$.23798531154657 = \frac{1}{4.20194}$$

7.30048531154657, the root is 2.7019410

adding .5

gives 3.2019410 as a fourth value of y .

Again,

7.0625

$$\begin{array}{r} .237985254909576312 \\ \hline \end{array} = \frac{1}{4.2019410}$$

7.300485254909576312, the root is 2.701941016

adding .5

gives 3.201941016 as a fifth value of y .

Presuming these figures are sufficiently correct to be made use of for obtaining the value of the root x , I take

$x = \sqrt{\left(\frac{y+1 \times 5}{y}\right)} = \sqrt{6.56155281281421331466} = 2.5615528128$, which eleven figures are all correct; and they have been obtained by employing considerably less than one-third of the figures necessary to the solution of the same equation by Raphson's method.

If x had been taken equal to $\frac{cy}{b}$, the whole of the eleven figures would have been obtained correct; and the last process, which is the most tedious of all, might have been dispensed with. The real value of the root x is $.5 + \sqrt{4.25}$; and the real value of the root y , by Rule VIII. is $\sqrt{\left(\frac{4.5 + \sqrt{4.25} \times 7.8125}{5}\right)}$.

The values of the roots x and y are always accurately discoverable, or undiscoverable, together.

There is somewhat worthy of notice in the values of x^2 and x ; for, excepting the integers, the figures are the same in each.

The equation given in this present example, is as difficult of solution as any that can be proposed.

Ex. III.

Let $x^3 - 15x = 4$.

$$\frac{b^3}{c^2} = 210.9375$$

the cotemporary equation is $y^3 - 210.9375y = 210.9375$

$$\sqrt{210.93} = 14.5$$

adding .4gives 14.9 as a first value of y .

Then,

$$210.9375$$

$$.75$$

$$210.1875$$

$$.0628 = \frac{1}{15.9}$$

$$210.2503, \text{ the square root is } 14.50$$

adding .5gives 15.00 as a second value of y .

Again,

$$210.9375$$

$$.75$$

$$210.1875$$

$$.0625 = \frac{1}{16}$$

$$210.25, \text{ the square root is } 14.5$$

adding .5gives 15 as a third value of y .

Hence the value of y is exactly 15; for $15^2 - 15 + \frac{15}{16}$, is exactly 210.9375.

The root of the original equation is, therefore, 4; for $x = \frac{cy}{b} = \frac{60}{15}$,
or $x = \sqrt{\left(\frac{y+1}{y} \times b\right)} = \sqrt{\left(\frac{240}{15}\right)} = 4$.

The reader will notice the manner of proving the correctness of the value of y , which is effected by taking the value of $y^2 - y + \frac{y}{y+1}$: if the value of this quantity had been less than 210.9375, it would have shewn that the approximating value of y was in defect; had it been greater, it would have shewn that the approximating value was in excess.

Ex. IV.

Let $x^3 - x = \frac{1}{3}$.

$$\frac{b^3}{c^2} = 9$$

the cotemporary equation is $y^3 - 9y = 9$
 $\sqrt{9 + .4} = 3.4$ as a first value of y .

Then,

$$\begin{array}{r} 9 \\ .75 \\ \hline 8.25 \\ .227272 = \frac{1}{4.4} \\ \hline \end{array}$$

8.477272, the square root is 2.911

adding .5

gives 3.411 as a second value of y .

Again,

$$\begin{array}{r} 8.25 \\ .2267059623 = \frac{1}{4.411} \\ \hline \end{array}$$

8.4767059623, the root is 2.91147

adding .5

gives 3.41147 as a third value of y .

Again,

$$\begin{array}{r} 8.25 \\ .22668180901150 = \frac{1}{4.41147} \\ \hline \end{array}$$

8.47668180901150, the root is 2.9114741

adding .5

gives 3.4114741 as a fourth value of y .

These eight figures are correct for an approximating value of y : consequently x is equal to $\sqrt{\left(\frac{y+1}{y} \times b\right)}$, or to $\frac{cy}{b}$, or to $\frac{y}{3} = 1.1371580 +$.

But on referring to page 271 of the Appendix to Mr. Frend's Algebra, we find the value of x stated as 1.137158164, which is erroneous: for

$$1.1371581^3 = 1.470491598 +$$

$$\text{subtracting } 1.1371581$$

$$\text{leaves } .333333498,$$

and shews that the value of x is not so great as 1.1371581.

$$x = 1.1371580426 +$$

This is the only instance of erroneous computation that I have discovered in approximations made by the skilful approximator and excellent algebraist, Baron Maseres ; and while I am mentioning his name, I should be deficient in my duty to my readers, were I not most earnestly to recommend to their perusal his " Dissertation on the Use of the Negative Sign," which, I venture to say, will give them a greater knowledge of that department of algebra than any other work whatever.

Ex. V.

Let $x^3 - 90x = 98$.

$$\frac{b^3}{c^2} = 75.9058725531828738 +$$

the cotemporary equation is

$$y^3 - 75.9058725531828738y = 75.9058725531828738$$

$$\sqrt{75.90 + .4} = 9.1 \text{ as a first value of } y;$$

or, by using the fractions, which are $\frac{9}{10}$ and $\frac{10}{11}$, and only using six figures of the coefficient, we get

$$y \text{ less than } \sqrt{\left(\frac{10 \times 75.9 \&c.}{9}\right)} = 9.18 +$$

$$y \text{ greater than } \sqrt{\left(\frac{11 \times 75.9 \&c.}{10}\right)} = 9.13 +:$$

therefore the figures 9.1 are correct for a first value of y .

A little more boldness than usual will be exercised in respect of this equation: yet, it must be observed, that two figures only, taken at a time, ensure correctness; whereas, if three or more are taken at a time, the convergency is hazarded.

Then, 75.90587255

.75

75.15587255

$$.09900990 = \frac{1}{10.1}$$

75.25488245, the root is 8.6749

adding .5

gives 9.1749 as a second value of y .

Again,

75.1558725531828738

$$.098281064187363020 = \frac{1}{10.1749}$$

75.254153617370236820, the root is 8.674915193

adding .5

gives 9.174915193 as a third value of y .

$a = \frac{cy}{b} = 9.99046321$: the last figure, however, is in excess; but by

taking $x = \sqrt{\left(\frac{y+1 \times 90}{y}\right)} = \sqrt{99.80935497569127231056} = 9.990463201$,

we get nine places correct, and perhaps the tenth also. Five, of the six examples contained in the Appendix to Mr. Frend's Algebra, have now been solved by the present method. The value of the root of the other equation, namely, $x^3 - 63x = 162$, would have appeared correct, by means of three of the rules.

Several other examples will be given, to exhibit the method in every necessary variety.

Ex. VI.

Let $x^3 - 800x = 8000$.

$$\frac{b^3}{c^2} = 8$$

the cotemporary equation is $y^3 - 8y = 8$

$\sqrt{8 + .4} = 3.2$ as a first value of y .

Then,

$$\begin{array}{r} 8 \\ .75 \\ \hline \end{array}$$

$$\begin{array}{r} 7.25 \\ .238095 = \frac{1}{4.2} \\ \hline \end{array}$$

7.488095, the square root of which is 2.736

adding .5

gives 3.236 as a 2d value of y .

Again,

$$\begin{array}{r} 7.25 \\ .2360717658 = \frac{1}{4.236} \\ \hline \end{array}$$

7.4860717658, the root is 2.73606

adding .5

gives 3.23606 as a third value of y .

Again,

$$\begin{array}{r} 7.25 \\ .23606842207145 = \frac{1}{4.23606} \\ \hline \end{array}$$

7.48606842207145, the root is 2.7360680

adding .5

gives 3.2360680 as a fourth value of y .

Again,

$$\begin{array}{r} 7.25 \\ .236067976245895958 = \frac{1}{4.236068} \\ \hline \end{array}$$

7.486067976245895958, the root is 2.736067977

adding .5

gives 3.236067977 as a fifth value of y .

These figures are certainly correct, for y is accurately equal to $1 + \sqrt{5}$, and $x = 32.36067977$, for $x = 10y$.

At the fourth value of y there was a small excess, but the subsequent operation corrected it.

A singularity occurs in this example: the value of $\frac{1}{y+1}$ has the same figures in its decimal places as y has.

Ex. VII.

Let $x^3 - 90x = 100$.

$$\frac{b^3}{c^2} = 72.9$$

the cotemporary equation is $y^3 - 72.9y = 72.9$

$\sqrt{72.9 + .4} = 8.9$ as a first value of y .

Then,

$$\begin{array}{r} 72.9 \\ .75 \\ \hline 72.15 \\ .101010 = \frac{1}{9.9} \\ \hline \end{array}$$

72.251010, the root is 8.500

adding .5

gives 9.000 as a second value of y .

Again,

$$\begin{array}{r} 72.15 \\ .1 = \frac{1}{10} \\ \hline \end{array}$$

72.25, the root is 8.5

adding .5

gives 9 as a third value of y .

And here we must necessarily stop, for we can get no other divisor than 10; and therefore 9 is the accurate value of y , $x = \frac{cy}{b} = \frac{900}{90} = 10$;

or $x = \sqrt{\left(\frac{y+1 \times b}{y}\right)}$

Ex. VIII.

Let $x^3 - 14x = \sqrt{15}$.

$$\frac{b^3}{c^2} = 182.9\dot{3}$$

the cotemporary equation is

$$y^3 - 182.9\dot{3}y = 182.9\dot{3}$$

the root would come out correct by the fractions; I therefore take $\sqrt{182 + .4} = 13.8$.

Then,

$$\begin{array}{r} 182.9333 \\ .75 \\ \hline \end{array}$$

$$\begin{array}{r} 182.1833 \\ .0675 = \frac{1}{14.8} \\ \hline \end{array}$$

182.2508, the root is 13.50

adding .5

gives 14.00 as a second value of y .

Again,

$$\begin{array}{r} 182.18333333 \\ .06666666 = \frac{1}{15} \\ \hline \end{array}$$

182.24999999, the root is 13.5, because the decimals repeat

adding .5

gives 14 as a third value of y .

This value is correct, and $x = \frac{cy}{b} = \frac{14\sqrt{15}}{14} = \sqrt{15}$.

If the equation $y^3 - 182.9\dot{3}y = 181.9\dot{3}$ had been solved, we should have got 13.9 for a first value of y .

Ex. IX.

Let $x^3 - 3x = 1$.

The cotemporary equation is $y^3 - 27y = 27$

$$\sqrt[3]{27 + .4} = 5.5.$$

Then,

$$\begin{array}{r} 27 \\ .75 \\ \hline 26.25 \\ .153846 = \frac{1}{6.5} \\ \hline \end{array}$$

26.403846, the root is 5.138

adding .5

gives 5.638 as a second value of y .

Again,

$$\begin{array}{r} 26.25 \\ .1506477854 = \frac{1}{6.638} \\ \hline \end{array}$$

26.4006477854, the root is 5.13815

adding .5

gives 5.63815 as a third value of y .

Again,

$$\begin{array}{r} 26.25 \\ .15064438134118 = \frac{1}{6.63815} \\ \hline \end{array}$$

26.40064438134118, the root is 5.1381557

adding .5

gives 5.6381557 as a fourth value of y .

Again,

$$\begin{array}{r} 26.25 \\ .15064425198703911087 = \frac{1}{6.6381557} \\ \hline \end{array}$$

26.40064425198703911087, the root is 5.138155724

adding .5

gives 5.638155724 as a fifth value of y .

Presuming on the accuracy of this value, I take $x = \frac{y}{3} = 1.879385241$, which is correct in every figure.

This example having much engaged the attention of mathematicians, the values of t and v , in the equation $3x - x^3 = 1$, will be given to a sufficient number of decimal places.

$$t \text{ and } v = .939626 \pm \sqrt{.35093345251572} :$$

using the sign $+$, $t = 1.5320889$

using the sign $-$, $v = .3472963$,

and all these figures are correct.

But in Dr. Hutton's Mathematical Dictionary, *Addenda et Corrigenda*, page 744, there are incorrect answers for the values of t and v , arising, as I conceive, from an erroneous computation in the tables of sines.

That the number .3472964, there given as the value of v , is erroneous, may be thus shewn :

$$3 \times .3472964 = 1.0418892$$

$$\text{subtracting } .3472964^3 = .041889082 +$$

leaves 1.000000118 :

and that the number 1.5320888, there given as the value of t , is erroneous, may be shewn by the correctness of our own number ;

$$\text{for } 3 \times 1.5320889 = 4.5962667$$

$$\text{subtracting } 1.5320889^3 = 3.596266755 +$$

there remains .999999945

Ex. X.

Let $x^3 - 10x = .00002$.

$$\frac{b^3}{c^2} = 2500000000000$$

the fractions are $\frac{1581139}{1581140}$ and $\frac{1581140}{1581141}$

and the cotemporary equation is

$$y^3 - 2500000000000y = 2500000000000$$

$$y \text{ is less than } \sqrt{\left(\frac{1581140 \times 2500 \text{ \&c.}}{1581139}\right)} = \sqrt{2500001581138.6601683975} \\ = 1581139.33008 +$$

$$y \text{ is greater than } \sqrt{\left(\frac{1581141 \times 2500 \text{ \&c.}}{1581140}\right)} = \sqrt{2500001581137.6601692512} \\ = 1581139.33008 + :$$

but as every figure for the value of y is the same in each limit, they are all assuredly correct; and it is probable that more figures would have been obtained, if the operation had been extended. It now remains to find the value of x , by means of the discovered value of y .

$$x = \frac{cy}{b} = \frac{y}{506000} = 3.162278660168, \text{ which value is true in every figure.}$$

This example deserves particular notice; for it serves to shew, that in the solution of an equation of the form $x^3 - bx = c$, when the value of $\frac{b^3}{c^2}$ is a high number, a sufficient value of the root may be found at a single step of the approximation.

Such equations are very easy of solution; and it is not necessary to give any more examples of this kind.

Ex. XI.

Let $x^3 - 15x = 14$.

$$\frac{b^3}{c^2} = 17.21938775510204 +$$

the cotemporary equation is

$$y^3 - 17.21938775510204y = 17.21938775510204$$

$$\sqrt{17.21 + .4} = 4.5.$$

Then,

$$17.219387$$

$$.75$$

$$16.469387$$

$$.181818 = \frac{1}{5.5}$$

$$16.651205, \text{ the root is } 4.080$$

adding .5

gives 4.580 as a second value of y .

Again,

$$17.2193877551$$

$$.75$$

$$16.4693877551$$

$$.1792114695 = \frac{1}{5.58}$$

$$16.6485992246, \text{ the root is } 4.08026$$

adding .5

gives 4.58026 as a third value of y .

Again,

$$17.21938775510204$$

$$.75$$

$$16.46938775510204$$

$$.17920311956790 = \frac{1}{5.58026}$$

$$16.64859087466994, \text{ the root is } 4.0802684$$

adding .5

gives 4.5802684 as a fourth value of y .

Presuming the work is correct, I take

$x = \sqrt{\frac{(y+1) \times 15}{y}} = \sqrt{18.27491725157416} = 4.2749172$, which value is correct in every figure; but if x had been taken equal to $\frac{cy}{b}$, only seven correct figures would have been obtained.

The value of x is accurately $.5 + \sqrt{14.25}$.

Ex. XII.

Let $x^3 - 27x = 1$.

$$\frac{b^3}{c^2} = 19683$$

the cotemporary equation is $y^3 - 19683y = 19683$
 $\sqrt{19683} + .4 = 140.6$ as a first value of y .

Then,

$$\begin{array}{r} 19683 \\ .75 \\ \hline 19682.25 \\ .007062 = \frac{1}{141.6} \\ \hline 19682.257062, \text{ the root is } 140.293 \\ \text{adding } .5 \end{array}$$

gives 140.793 as a second value of y .

Again,

$$\begin{array}{r} 19682.25 \\ .0070525343 = \frac{1}{141.793} \\ \hline 19682.2570525343, \text{ the root is } 140.29346 \\ \text{adding } .5 \end{array}$$

gives 140.79346 as a fourth value of y .

Presuming these figures to be correct, I take

$x = \sqrt[3]{\frac{y+1 \times 27}{y}} = 5.2145728$, which figures are all correct; but if x had been taken equal to $\frac{cy}{b}$, the last resulting figure would have been incorrect.

There is a necessary remark to be made in this place; namely, that when the coefficient of the cotemporary equation is a very high number, as it is in the present case, and that the value of the root y consists of three or more integers, they must all be taken, together with the first decimal, if any, for the first value of y , adding .4 thereto, otherwise the decimals resulting from the quantity $\frac{1}{y+1}$ would be incorrectly placed.

Ex. XIII.

Let $x^3 - 16x = 24$, or let $w^3 - w = \frac{24}{16\sqrt{16}}$.

$$\frac{b^3}{c^2} = 7.1$$

the cotemporary equation is $y^3 - 7.1y = 7.1$

$$\sqrt{7+.4} = 3 +$$

Here one figure only will be taken, for the sake of an observation about to be made at the next step.

Then,

7.1111

.75

6.3611

.2499 = $\frac{1}{4}$ - .0001, and this subtraction is made, because the value of

6.6110, the root is 2.57

adding .5

gives 3.07 as a second value of y .

Again,

6.36111111

.24570024 = $\frac{1}{4.07}$

6.60681135, the root is 2.5703

adding .5

gives 3.0703 as a third value of y .

Here it was necessary to take a fifth figure, because a cypher appeared; the work could not have been continued without a new figure.

Again,

6.3611111111

.2456821364 = $\frac{1}{4.0703}$

6.6067932475, the root is 2.57036

adding .5

gives 3.07036 as a fourth value of y .

Only one figure has been taken, in order to adhere to the plan on which the preceding accurate results have been found.

Again,

$$\begin{array}{r} 6.36111111111111 \\ .24567851492251 = \frac{1}{4.07036} \\ \hline \end{array}$$

6.60678962603362, the root is 2.5703676

adding .5

gives 3.0703676 as a fifth value of y .

Presuming these figures to be correct, I take

$$x = \sqrt{\left(\frac{y+1}{y} \times 16\right)} = \sqrt{21.21110241001761} = 4.6055512; \text{ and these figures}$$

are all correct, for $x = 1 + \sqrt{13}$.

$$\text{In the second original equation, } w = \sqrt{\left(\frac{y+1}{y}\right)}.$$

Ex. XIV.

Let $x^3 - 98x = 20$.

This equation may be solved by Rule XV.:

$$\text{for } x^2 + 1 = \sqrt{(b+2 \times x^2 + cx + 1)}$$

$$x^2 + 1 = \sqrt{(100x^2 + 20x + 1)}$$

$$x^2 + 1 = 10x + 1$$

$$x = 10.$$

It may be also solved accurately by means of the fractions, and by the method prescribed in Rule XIX., but it is now to be done by approximation.

$$\frac{b^3}{c^2} = 2352.98,$$

the cotemporary equation is $y^3 - 2352.98y = 2352.98$

$$\sqrt{2352.98 + .4} = 48.9 +.$$

Then,

2352.98

.75

2352.23

$$.020020 = \frac{1}{49.9}$$

2352.250020, the root is 48.500

adding .5

49 as a second value of y

Again,

2352.23

$$.02 = \frac{1}{50}$$

2352.25, the root is 48.5

adding .5

gives 49 as a third value of y .

And here we must necessarily stop, because no new divisor can be obtained. It follows, that 49 is the exact value of y ; and, taking

$$x = \frac{cy}{b} = \frac{980}{98} = 10, \text{ the accurate value of } x \text{ is discovered.}$$

Ex. XV.

$$\text{Let } x^3 - 40x = 24$$

$$w^3 - 10w = 3$$

$$z^3 - .1z = .003$$

$$\frac{b^3}{c^2} = 111.\dot{1}, \text{ in all these equations;}$$

$$\text{the cotemporary equation is } y^3 - 111.\dot{1}y = 111.\dot{1}$$

$$\sqrt{111.11 + .4} = 10.9 \text{ as a first value of } y.$$

Then,

$$111.111111$$

$$.75$$

$$\begin{array}{r} 110.361111 \\ .084033 = \frac{1}{11.9} \end{array}$$

$$110.445144, \text{ the root is } 10.509$$

$$\text{adding } .5$$

gives 11.009 as a second value of y .

Two figures were taken at the preceding step, because the work could not have been continued without the accession of the last figure.

Again,

$$110.36111111$$

$$.08327088 = \frac{1}{12.009}$$

$$110.44438199, \text{ the root is } 10.5092$$

$$\text{adding } .5$$

gives 11.0092 as a third value of y .

Again,

$$110.361111111111$$

$$.083269493388 = \frac{1}{12.0092}$$

$$110.444380604499, \text{ the root is } 10.509252$$

$$\text{adding } .5$$

gives 11.009252 as a fourth value of y .

$x = \sqrt{\left(\frac{y+1 \times 40}{y}\right)} = \sqrt{43.63330769429203} = 6.6055512$, which is correct in every figure.

$$w = \frac{x}{2} = 3.3027756$$

$$z = \frac{w}{10} = .33027756$$

Ex. XVI.

Let $x^3 - 9x = 10$.

This equation was proposed for solution by *Cardan* to *Tartaglia*.

$$\frac{b^3}{c^2} = 7.29$$

the cotemporary equation is $y^3 - 7.29y = 7.29$

$\sqrt{7.29} + .4 = 3.1$ as a first value of y .

Then,

7.29

.75

6.54

$$.243902 = \frac{1}{4.1}$$

6.783902, the root of which is 2.604

adding .5

gives 3.104 as a second value of y .

Again,

6.54

$$.2436647173 = \frac{1}{4.104}$$

6.7836647173, the root is 2.60454

adding .5

gives 3.10454 as a third value of y .

Again,

6.54

$$.24363266041992 = \frac{1}{4.10454}$$

6.78363266041992, the root is 2.6045407

adding .5

gives 3.1045407 as a fourth value of y .

These figures are all correct, and taking $x = \sqrt{\left(\frac{y+1}{y} \times 9\right)} = 3.4494897$;

these figures are also correct.

If the operation had been continued without a new divisor, one if not two more correct figures would have been obtained, for it has been tried.

The roots of both these equations may be correctly found; and as they are connected with the ancient history of our case, it will be shewn how they may be discovered.

Take, by Rule IX., $\frac{4}{9}$ of the coefficient of each equation for d , and the remaining $\frac{5}{9}$ ths for e .

$$\begin{aligned} \text{Then, } d &= 4, & e &= 5 & \text{for the case } x, \\ d &= 3.24, & e &= 4.05 & \text{for the case } y, \\ x &= 1 + \sqrt{6}, & y &= .9 + \sqrt{4.86}, \end{aligned}$$

and there are two other real expressions for the values of these roots.

$$t = 2, \quad v = \sqrt{6} - 1 \text{ for the case } x.$$

There is no other equation of the form $x^3 - bx = b + 1$, b being a number consisting of integers only, that can be accurately solved, the equation being in the irreducible case.

Ex. XVII.

Let $x^3 - .8x = .02$

$$w^3 - \sqrt[3]{1280}w = 1$$

$$\frac{b^3}{c^2} = 1280$$

the cotemporary equation is $y^3 - 1280y = 1280$

$$\sqrt[3]{1280} + .4 = 36.1.$$

Then,

1280

.75

1279.25

$$.0269 = \frac{1}{37.1}$$

1279.2769, the root of which is 35.76

adding .5

gives 36.26 as a second value of y .

Again,

1279.25

$$.02683843 = \frac{1}{37.26}$$

1279.27683843, the root is 35.7669

adding .5

gives 36.2669 as a third value of y .

Again,

1279.25

$$.026833463475 = \frac{1}{37.2669}$$

1279.276833463475, the root is 35.766979

adding .5

gives 36.266979 as a fourth value of y .

Presuming these figures to be correct, I take $x = \sqrt{\left(\frac{y+1}{y} \times .8\right)} =$

$\sqrt{.8220586335575400} = .90667449$, which value is correct in every figure; for $x^3 - .8x = .019999997 +$, using the discovered value.

The value of the root w may be discovered to as many figures, by taking w equal to $\sqrt{\left(\frac{y+1}{y} \times \sqrt[3]{1280}\right)}$.

Neither of these three equations has accurately discoverable roots.

Ex. XVIII.

Let $x^3 - 16.25x = 1$.

$$\frac{b^3}{c^2} = 4291.015625$$

the cotemporary equation is $y^3 - 4291.015625y = 4291.015625$
 $\sqrt{4291.01} + .4 = 65.9$ as a first value of y .

Then,

$$\begin{array}{r} 4291.0156 \\ .75 \\ \hline 4290.2656 \\ .0149 = \frac{1}{66.9} \\ \hline \end{array}$$

4290.2805, the root of which is 65.5002

adding .5

gives 66.0002 as a second value of y .

All these new places were necessarily taken, on account of the appearance of two cyphers; the work could not have been continued without a new final figure.

Again,

$$\begin{array}{r} 4290.265625 \\ .014925328581 = \frac{1}{67.0002} \\ \hline \end{array}$$

4290.280550328581, the root is 65.500233

adding .5

gives 66.000233 as a third value of y :

and this value is correct in every figure, for $y = \frac{65 + \sqrt{4489.0625}}{2}$

$x = \sqrt{\left(\frac{y+1}{y} \times 16.25\right)} = 2 + \sqrt{4.25} = 4.06155281 +$, of which every figure is correct.

In this example the coefficient is divided into sixty-five equal parts, whereof sixty-four are taken for d ; and the remaining part, namely, .25, is taken for e , by Rule IX.

Ex. XIX.

Let $x^3 - 16.5x = 2$.

$$\frac{b^3}{c^2} = 1123.03125$$

the cotemporary equation is $y^3 - 1123.03125y = 1123.03125$
 $\sqrt[3]{1123.03 + .4} = 33.9$ as a first value of y .

Then,

1123.03125

.75

1122.28125

.02865 = $\frac{1}{34.9}$

1122.30990, the root of which is 33.5008

adding .5

gives 34.0008 as a second value of y .

Three places of decimals were taken at this last step, for the reason given in the last example.

Again,

1122.28125

.028570775525 = $\frac{1}{35.0008}$

1122.309820775525, the root of which is 33.500892

adding .5

gives 34.000892 as a third value of y .

These figures are all correct, for the exact value of y is

$$\frac{33 + \sqrt{1225.125}}{2} = 34.000892 +.$$

$x = \sqrt{\left(\frac{y+1}{y} \times 16.5\right)} = 2 + \sqrt{4.5} = 4.12132034 +$, which figures are also correct.

In this example the coefficient is divided into thirty-three equal parts, whereof thirty-two parts are taken for d , and the remaining part for e , by Rule IX.

Ex. XX.

Let $x^3 - 280x = 792$.

$$\frac{b^3}{c^2} = 34.99641299357208 +$$

the cotemporary equation is

$$y^3 - 34.99641299357208y = 34.99641299357208$$

$\sqrt{34.99 + .4} = 6.3$ as a first value of y .

Then,

$$\begin{array}{r} 34.996412 \\ .75 \\ \hline 34.246412 \\ .136986 = \frac{1}{7.3} \\ \hline \end{array}$$

34.383398, the root of which is 5.863

adding .5

gives 6.363 as a second value of y .

Again,

$$\begin{array}{r} 34.2464129935 \\ .1358142063 = \frac{1}{7.363} \\ \hline \end{array}$$

34.3822271998, the root is 5.86363

adding .5

gives 6.36363 as a third value of y .

Again,

$$\begin{array}{r} 34.24641299357208 \\ .13580258649606 = \frac{1}{7.36363} \\ \hline \end{array}$$

34.38221558006814, the root is 5.8636350

adding .5

gives 6.3636350 as a fourth value of y .

Prestuming the work to be correct, I take

$$x = \sqrt{\frac{y+1 \times 280}{y}} = \sqrt{324.000009428573} = 18.000000 +, \text{ which is correct; for } x \text{ is exactly } 18.$$

If the work had been continued, more cyphers would have been obtained. The student is requested to give particular attention to this example, and to take the value of $\frac{b^3}{c^2}$ to a greater extent, in order to continue the operation.

Ex. XXI.

Let $x^3 - 9x = 8$.

$$\frac{b^3}{c^2} = 11.390625$$

the cotemporary equation is $y^3 - 11.390625y = 11.390625$

$\sqrt{11.39 + .4} = 3.7$ as a first value of y .

Then,

11.390625

.75

10.640625

.212765 = $\frac{1}{4.7}$

10.853390, the root of which is 3.294

adding .5

gives 3.794 as a second value of y .

Again,

10.640625

.2085940758 = $\frac{1}{4.794}$

10.8492190758, the root is 3.29381

adding .5

gives 3.79381 as a third value of y .

The three new figures are correct. The final figure at the first step was in excess.

Again,

10.640625

.20860234343872 = $\frac{1}{4.79381}$

10.84922734343872, the root is 3.2938165

adding .5

gives 3.7938165 as a fourth value of y .

Presuming that these figures are correct, I take $x = \frac{cy}{b} = \frac{8 \times 3.7938165}{9} = 3.3722818 +$, which figures are all accurate; for $x = .5 + \sqrt{8.25}$, or, by a second expression, $x = \frac{\sqrt{8.5 - \sqrt{8.25}} + \sqrt{10.5 + 3\sqrt{8.25}}}{2}$

It is of considerable use to know, that there are always two expressions equivalent to x , whereof one is much less complex than the other, and consequently more easily brought into figures.

The singularity noticed in Ex. III. occurs in this present example; that is, the decimals of x^2 and x are the same.

Ex. XXII.

Let $x^3 - 7x = 1$.

$$\frac{b^3}{c^2} = 343$$

the cotemporary equation is $y^3 - 343y = 343$

$$\sqrt[3]{343} + .4 = 18.9.$$

Then,

343

.75

342.25

$$.0502 = \frac{1}{19.9}$$

342.3002, the root of which is 18.501

adding .5.

gives 19.001 as a second value of y .

The appearance of a cypher made it necessary to take the last figure.

Again,

342.25

$$.04999750 = \frac{1}{20.001}$$

342.29999750, the root is 18.5013

adding .5

gives 19.0013 as a third value of y .

Again,

342.25

$$.049996750211 = \frac{1}{20.0013}$$

342.299996750211, the root is 18.501351

adding .5

gives 19.001351 as a fourth value of y .

Presuming the work to be correct, I take $x = \sqrt{\left(\frac{y+1}{7} \times 7\right)}$, or
 $x = \frac{y}{7} = 2.7144787$, which is true in every figure; for $x^3 - 7x =$
 $.9999993 +$, using the discovered value.

Er. XXIII.

Let $x^3 - 77284x = 8013128$.

This equation was formerly solved by the learned Mr. Raphson, and is on that account peculiarly worthy of the attention of the reader.

$$\frac{b^3}{c^2} = 7.18893596637029 +$$

the cotemporary equation is

$$y^3 - 7.18893596637029y = 7.18893596637029$$

$$\sqrt{7.18 + .4} = 3 + \text{ as a first value of } y.$$

Then,

7.18

.75

6.43

$$.24 = \frac{1}{4} - .01$$

6.67, the root of which is 2.58

adding .5

gives 3.08 as a second value of y .

Again,

6.438935

$$.245098 = \frac{1}{4.08}$$

6.684033, the root is 2.585

adding .5

gives 3.085 as a third value of y .

Again,

6.4389359663

$$.2447980416 = \frac{1}{4.085}$$

6.6837340079, the root is 2.58529

adding .5

gives 3.08529 as a fourth value of y .

Again,

6.43893596637029

$$.24478066428576 = \frac{1}{4.08529}$$

6.68371663065605, the root is 2.5852885

adding .5

gives 3.0852885 as a fifth value of y .

K

Presuming the work is correct, I take

$$x = \sqrt{\left(\frac{y+1 \times 77284}{y}\right)} = \sqrt{102333.1971820463} = 319.89560, \text{ and these}$$

figures and the cypher are all accurately true; for

$$319.89560^3 = 32735938.782247 +$$

$$\text{subtracting } 77284 \times x = 24722811.5504$$

$$\text{there remains } 8013127.231847$$

The values of the roots t and v will now be given by means of the discovered value of x ; and for this purpose Rule IV. will be employed, and the equation $xt - t^2 = x^2 - b$ solved.

$$t \text{ and } v = \frac{319.89560 \pm \sqrt{102333.19489936 - 100196.77959744}}{2}$$

$$= 159.9478 \pm \sqrt{534.10382548}$$

$$\text{using the sign } +, t = 183.058486$$

$$\text{using the sign } -, v = 136.837114$$

$$319.89560$$

See Part II. of Friend's Principles of Algebra, pages 44, 45; edit. 1796.

Ex. XXIV.

Let $x^3 - 1000x = 174$.

See Friend's Algebra, Part II. page 42, where this equation is quoted from Raphson.

$$\frac{b^3}{c^2} = 33029.4622803540$$

the cotemporary equation is

$$y^3 - 33029.4622803540y = 33029.4622803540$$

$$\sqrt{33029.46 + .4} = 182.1 \text{ as a first value of } y.$$

Then,

$$33029.462280$$

$$.75$$

$$33028.712280$$

$$.005461 = \frac{1}{183.1}$$

$$33028.717741, \text{ the root is } 181.738$$

$$\text{adding } .5$$

$$\text{gives } 182.238 \text{ as a second value of } y.$$

Again,

$$33028.7122803540$$

$$.0054573832 = \frac{1}{183.238}$$

$$33028.7177377372, \text{ the root is } 181.73804$$

$$\text{adding } .5$$

$$\text{gives } 182.23804 \text{ as a third value of } y.$$

Presuming the preceding work to be correct, I take

$$x = \sqrt{\frac{y+1 \times 1000}{y}} = \sqrt{1005.487328551163} = 31.709420: \text{ and these}$$

figures and the cypher are certainly correct; for

$$31.709421^3 = 31883.422617 +$$

$$\text{subtracting } 31.709421 \times 1000 = 31709.421$$

$$\text{there remains } 174.001647 +,$$

which shews that 31.709421 is too great for the value of x , although it is exceedingly near the truth.

There can be no other number accurately true to eight places for such value, except 31.709420.

But how can this solution be reconciled with that given by Raphson, as quoted by Mr. Friend, and confirmed by logarithms?

In page 43 of Mr. Friend's Algebra, Part II.

the root v is said to be .174005268

the root t is said to be 31.5363

making 31.7103+ :

but both these values cannot be correct, for if the number 31.71 be cubed, it will be found too great for the value of x .

The values of the roots t and v will now be given.

t and $v = 15.854710 \pm \sqrt{245.8845124477}$

using the sign +, $t = 31.535415$

using the sign -, $v = .174005$

giving $31.709420 = x$.

Ex. XXV.

Let $x^3 - 13x = 1$

$$\frac{b^3}{c^2} = 2197$$

the cotemporary equation is $y^3 - 2197y = 2197$ $\sqrt{2197} + .4 = 47.2$ as a first value of y .

Then,

$$\begin{array}{r} 2197 \\ .75 \\ \hline \end{array}$$

$$\begin{array}{r} 2196.25 \\ .0207 = \frac{1}{48.2} \\ \hline \end{array}$$

2196.2707, the root of which is 46.86

adding .5

gives 47.36 as a second value of y .

Again,

$$\begin{array}{r} 2196.25 \\ .02067824 = \frac{1}{48.36} \\ \hline \end{array}$$

2196.27067824, the root is 46.8643

adding .5

gives 47.3643 as a third value of y .

Again,

$$\begin{array}{r} 2196.25 \\ .020676408011 = \frac{1}{48.3643} \\ \hline \end{array}$$

2196.270676408011, the root is 46.864386

adding .5

gives 47.364386 as a fourth value of y .

Presuming the preceding arithmetical operation to have been correctly performed, I take $x = \frac{cy}{b} = \frac{y}{13} = 3.6434143$, which is true in

every figure, for $3.6434143^3 = 48.364385666 +$

subtracting $13 \times 3.6434143 = 47.3643859$

there remains .999999766 +.

Ex. XXVI.

Let $x^3 - 400x = 400$.

$$\frac{b^3}{c^2} = 400$$

$\sqrt{400 + .4} = 20.4$ as a first value of x .

Then,

400

.75

399.25

$$.0467 = \frac{1}{21.4}$$

399.2967, the root of which is 19.98

adding .5

gives 20.48 as a second value of x .

Again,

399.25

$$.04655493 = \frac{1}{21.48}$$

399.29655493, the root is 19.9824

adding .5

gives 20.4824 as a third value of x .

Again,

399.25

$$.046549733735 = \frac{1}{21.4824}$$

399.296549733735, the root is 19.982406

adding .5

gives 20.482406 as a fourth value of x .

All these figures are correct, for $20.482406^3 = 8592.9623963 +$
 subtracting $400 \times 20.482406 = 8192.9624$

there remains 399.9999963 +

All the examples hitherto solved, belong to the irreducible case. The four examples which follow can be solved by the rule commonly known by the appellation of *Cardan's Rule*.

Ex. XXVII.

Let $x^3 - 6x = 6$.

This is an equation of the form $x^3 - bx = b$, and can be solved accurately by Cardan's rule, because $\frac{b^3}{b^2} = \frac{b^3}{c^2}$ is less than 6.75.

It is necessary to make one remark, previously to the solution of this equation. It will be shewn in the next example, that the approximating root of the equation $x^3 - 3x = 3$, is 2.103803+ : but $\sqrt[3]{3} = 1.7320508$, is less than 2.103803+ by the quantity .37+ ; and it has been shewn in Rule XVIII. that the root of the equation $x^3 - 6.75x = 6.75$, only exceeds the value of $\sqrt[3]{6.75} + .4$ by the very small quantity .001+. Therefore, when an equation of the form $x^3 - bx = b$ is proposed for solution, b being a number greater than 3, and less than 6.75, the quantity $.37 + \sqrt[3]{b}$ may be safely taken as an approximating value of x ; for x will necessarily be less than $\sqrt[3]{b} + .4002$. That the quantity $\sqrt[3]{b} + .37$ may be thus applied as a value of x , is evident from the known properties of cube numbers.

The equation $x^3 - 6x = 6$ will be solved on this principle, according to the general method.

$$\sqrt[3]{6} + .37 = 2.81 \text{ as a first value of } x.$$

Then,

6

.75

5.25

$$.262467 = \frac{1}{3.81}$$

5.512467, the root of which is 2.347

adding .5

gives 2.847 as a second value of x , and correct.

Again,

5.25

$$.2599428125 = \frac{1}{3.847}$$

5.5099428125, the root is 2.34732

adding .5

gives 2.84732 as a third value of x .

Again,

5.25

$$\frac{.25992119189461}{3.84732} = \frac{1}{3.84732}$$

5.50992119189461, the root is 2.3473221

adding .5

gives 2.8473221 as a fourth value of x .

All these figures are correct ;

for their cube = 23.083932565 +

subtracting $6x = 17.0839326$

there remains 5.999999965

The nearer value of the root x , by a subsequent operation, appeared to be 2.847322104, which is probably correct ; but the accuracy of these two additional places is not vouched for.

Having now a correct value of the root x to eight places, any equation of the form $y^3 - by = c$, the value of $\frac{b^3}{c^2}$ being exactly 6, can be solved to the same accuracy ; for $y = \sqrt{\left(\frac{x+1}{x} \times b\right)}$.

Ex. XXVIII.

Let $x^3 - 3x = 3$.

This equation is not in the irreducible case, because the value of $\frac{b^3}{c^2} = 3$ is less than 6.75. The method of approximation contained in this work is not adapted to solve equations of this form where the value of $\frac{b^3}{c^2}$ is less than 3; nor do I absolutely vouch for the accuracy of results where the value of $\frac{b^3}{c^2}$ is less than 5, which is very nearly the limit mentioned in the Introduction.

In the present equation, therefore, and in this equation alone, I am under the necessity of making an assumption.

Let, then, $\sqrt{3} + .37 = 2.1$ be assumed as the first near value of x .

Then,

$$\begin{array}{r} 3 \\ .75 \\ \hline 2.25 \\ .322258 = \frac{1}{3.1} \\ \hline 2.572258, \text{ the root is } 1.603 \\ \text{adding } .5 \end{array}$$

gives 2.103 as a second value of x .

Again,

$$\begin{array}{r} 2.25 \\ .32226877 = \frac{1}{3.103} \\ \hline 2.57226877, \text{ the root is } 1.6038 \\ \text{adding } .5 \end{array}$$

gives 2.1038 as a third value of x .

Again,

$$\begin{array}{r} 2.25 \\ .322185707842 = \frac{1}{3.1038} \\ \hline 2.572185707842, \text{ the root is } 1.603803 \\ \text{adding } .5 \end{array}$$

gives 2.103803 as a fourth value of x .

These figures are all correct; but they have not been obtained with the usual facility.

If the reader will endeavour to solve the equation $x^3 - .2x = .2$ by this method, he will meet with much greater difficulty; and I have reason to believe, that the solution is impracticable by any rule given in this work.

Fearing to be misunderstood, I think it necessary to observe again, that the method is very far from being so well adapted to equations which fall without the irreducible case, as it is to those which fall within it; and therefore, when great accuracy is required, the results obtained by this method for the roots of the former equations, should be contrasted with those obtained by Cardan's rule.

Ex. XXIX.

Let $x^3 - 74x = 260$.

$$\frac{b^3}{c^2} = 5.9944363905325$$

the cotemporary equation is

$$y^3 - 5.9944363905325y = 5.9944363905325$$

$$\sqrt{5.99 + .37} = 2.77 \text{ as a first value of } y.$$

Then,

5.994436

.75

5.244436

.265251 = $\frac{1}{3.77}$

5.509687, the root is 2.347

adding .5

gives 2.847 as a second value of y .

Again,

5.9944363905

.75

5.2444363905

.2599428125 = $\frac{1}{3.847}$

5.5043792030, the root is 2.34614

adding .5

gives 2.84614 as a third value of y .

Taking $x = \frac{cy}{b} = \frac{260 \times 2.84614}{74} = 9.99994$, a sufficiently approxi-

imating value is obtained, for x is accurately equal to 10. By proceeding in the same manner, a more considerable approximation would be derived.

Ex. XXX.

Let $x^3 - 47x = 136$.

$$\frac{b^3}{c^2} = 5.613267733 +$$

the cotemporary equation is $y^3 - 5.613267733y = 5.613267733$
 $\sqrt{5.6132 + .37} = 2.7 +$ as a first value of y .

Then,

5.613267

.75

4.863267

.270270 = $\frac{1}{3.7}$

5.133537, the root is 2.265

adding .5

gives 2.765 as a second value of y .

Again,

4.863267733

.2656042496 = $\frac{1}{3.765}$

5.1288719826, the root is 2.26470

adding .5

gives 2.76470 as a third value of y .

Taking $x = \frac{cy}{b} = \frac{136 \times 2.7647}{47} = 7.99998 +$, the approximation is as

considerable as could possibly take place with the few figures which were used, for x is accurately equal to 8.

The examples being concluded, it may be proper to shew how the proof of the accuracy of the value of y may be obtained without cubing the number. It has been shewn in Rule III., that in an equation of the form $y^3 - by = b$, $y^2 - y + \frac{y}{y+1}$ is equal to b : taking the discovered value of y in the second equation given for solution in this present example,

then $2.7647^2 = 7.64356609$

subtract 2.7647

4.87886609

adding $\frac{y}{y+1} = .734374$

gives 5.61324 +.

EQUATIONS

In the IRREDUCIBLE CASE, with correct approximating Answers, for the Student's Practice.

Let $x^3 - 3x=1$	Answer $x=1.879385241$
$x^3 - 4x=1$	$x=2.1149075$
$x^3 - 5x=1$	$x=2.3300587$
$x^3 - 6x=1$	$x=2.528917957$
$x^3 - 7x=1$	$x=2.7144787$
$x^3 - 8x=1$	$x=2.8889694$
$x^3 - 9x=1$	$x=3.0540842$
$x^3 - 10x=1$	$x=3.2111393$
$x^3 - 11x=1$	$x=3.36117758$
$x^3 - 12x=1$	$x=3.5050397$
$x^3 - 13x=1$	$x=3.6434143$
$x^3 - 14x=1$	$x=3.7768729$
$x^3 - 15x=1$	$x=3.9058959$
$x^3 - 16x=1$	$x=4.0308912$
$x^3 - 17x=1$	$x=4.1522085$
$x^3 - 18x=1$	$x=4.2701503$
$x^3 - 19x=1$	$x=4.3849801$
$x^3 - 20x=1$	$x=4.4969293$
$x^3 - x = .273$	$x=1.1156603957$
$x^3 - 2x = .573$	$x=1.540143877$
$x^3 - 3x=1.125$	$x=1.895643923$
$x^3 - 4x=2.625$	$x=2.270690632$
$x^3 - 5x=4.125$	$x=2.570027472$

Find the roots t and v of the opposite equations, $bx - x^3 = c$,

$$t \text{ and } v = \frac{x}{2} \pm \sqrt{\frac{4b - 3x^2}{2}}$$

the sign $+$ belonging to t .

time that the semi-circumference B E D is generated. Let the exterior point F be supposed to generate the curve F I H in the same time.

Under these circumstances, let the progress of the motion be supposed to stop at A M L K; then will the line K M, bounded by the exterior and interior curves, represent the root x of the equation $x^3 - bx = c = \text{chord L B}$; and the lines K L, L M, meeting at the semi-circumference at the point L, will respectively represent the two roots t and v of the equation $bx - x^3 = c = \text{chord L B}$: and a similar representation will take place at whatever situation the progress may be supposed to stop, the absolute term, equal to the chord at such situation, increasing or decreasing accordingly.

Again, let the progress be supposed to arrive at A D H; there the two roots t and v are equal, being each equal to radius, the line A H $= x$ being equal to their sum; and that situation is precisely the boundary of the irreducible case: but we know that the boundary is not included in the case.

Whatever numerical magnitude may be assigned to the line B F $= \sqrt{b}$, the diagram can undergo no alteration; for the ratios which it comprehends, are constant under all possible change of numerical magnitude that may be assigned to the line B F.

The line B F is the parent, if it may be so termed, of this case, and it represents the equations $x^3 - bx = 0$, or $bt - t^3 = 0$; so that both x and t are equal to the line B F $= \sqrt{b}$.

The length of the line B F is that of the chord of 120° of a circle, of which A B is radius.

The irreducible case is now about to commence, for at the instant A B F moves forward under the circumstances before mentioned, the root v comes into existence.

It has been usual with writers on trigonometry to consider radius as unity, and a happy coincidence attended the assumption; namely, that at the first boundary alluded to, the value of the root x and of the absolute term were each equal to 2, or to the length of the diameter. The same assumption may be now used. If, then, radius is called unity, B F is equal to $\sqrt{3}$, and the equations, represented by the line B F, are $x^3 - 3x = 0$, $3t - t^3 = 0$; so that x and t being each equal to $\sqrt{3}$, are equal to each other. At the instant of motion the irreducible case begins; for x has received an increase of magnitude; t has lost part of its magnitude, being no longer equal to $\sqrt{3}$; v is

generated, and is accurately equal to the sum of the increase of x , and of the decrease of t .

If the increase of x is called z , and if the decrease of t is called w , then v is equal to z and w , and the following equations result :

$$z^3 + 3\sqrt{bz^2} + 2bz = c.$$

$$w^3 - 3\sqrt{bw^2} + 2bw = c.$$

Here occurs the first difficulty of our case ; for, in order to obtain the numerical value of v , without which neither x nor t can be accurately found, the roots of both these equations must be obtained : but this cannot be generally effected ; for by exterminating the second terms, the equation $x^3 - bx = c$ is constantly produced.

Returning to the equations $x^3 - 3x = 0$, $3t - t^3 = 0$, let the increase of x be supposed equal to .1, then the following equation arises :

$$x^3 - 3x = .03\sqrt{3} + .601$$

$$\text{where } x = \sqrt{3} + .1 :$$

but in the equation $3x - x^3 = .03\sqrt{3} + .601$, the values of t and v are,

$$\frac{x}{2} \pm \frac{\sqrt{4b - 3x^2}}{2}$$

the sign $+$ belonging to t .

Hence the increase of x is considerably less than the decrease of t , and v is equal to the sum of such increase and decrease. But there is a limit to the decrease of t and to the magnitude of v , which limit takes place when they become equal to each other ; and beyond this limit their existence is impossible. There is, however, no limit to the magnitude of x .

The equations under consideration, are supposed to have the constant coefficient 3, equal to the square of the chord of 120° ; the limit beyond which t and v cannot exist, takes place when the equations become $x^3 - 3x = 2$, $3x - x^3 = 2$, for then t and v are each equal to unity ; $H D$ being equal to t , and $A D$ being equal to v ; and this is the only place in the diagram where they can conjunctively exist without appertaining to the irreducible case.

In this state, the whole increase which x has received, is $2 - \sqrt{3}$; the decrease which t has experienced, is $\sqrt{3} - 1$; and v has risen from an infinitely small quantity to 1. The curve $F I H$ has approached the semi-circumference in the ratio of the decrease of t ; the curve $B G A$ has continually approached the fixed point A , and ultimately fallen into it, according to the combined ratios of the decrease of t and of the increase of x . But there is another curve

deserving the attention of the reader, although it is not introduced into the diagram. This curve may be supposed to originate at the point B, to be formed by the continual union of z and w , and to separate the root v into two parts.

These three transcendental curves might be drawn with all possible accuracy, but I see no particular use it would lead to. The diagram itself might have been well spared in this work; and indeed it has been introduced only for the sake of making the reader familiar with the constitution of the case in question. It may be observed, that when the root t is perpendicular to the diameter, it is equal to the chord of 90° , and the value of $\frac{b^3}{c^2}$ is exactly 13.5: consequently, one half of the equations of the form $x^3 - bx = c$, while in the irreducible case, have their values of $\frac{b^3}{c^2}$ less than 13.5; and these equations are the most difficult of solution: the roots t and v continually approach nearer to each other in value, as the value of $\frac{b^3}{c^2}$ approaches its limit 6.75.

The difficulty decreases as the equations approach the point B; for the value of v decreases rapidly, and ultimately vanishes: and yet a great facility of solution does not take place until the absolute term, or chord by which the absolute term is represented, decreases to nearly 20° ; for at the chord of 60° , where the equation becomes $x^3 - 3x = 1$, and where the value of $\frac{b^3}{c^2}$ is 27, the solution is effected

with only a little more ease than where the value of $\frac{b^3}{c^2}$ is only 13. On referring to the diagram, and applying radius from B to the circumference at N, B N is the chord of 60° ; G O is equal to x , N O to t , and G N to v , which are the roots of the equations $x^3 - 3x = 1$ and $3x - x^3 = 1$, or of any other equations of these forms where the values of $\frac{b^3}{c^2}$ are equal to 27.

Perhaps the reader would wish to know my opinion on the possibility, or probability, of a reduction of the case.

The impossibility of a reduction has not been proved hitherto, and I think the epithet *irreducible* improperly applied to the case: it may therefore be possible; but the probability is not great. There is no Mecænas to encourage men in such arduous undertakings; and even some societies, from which we might expect the most brilliant discoveries to emanate, have, in consequence of incapacity of councils,

or inattention to science, degenerated into emporiums of ease and indolence.

I should not notice here, were it not for the public good, that I wrote a letter to the Secretary of the Royal Society of London, offering to communicate the method of approximation which I have given in this book, and requesting to know if such communication would be agreeable. My letter reached the Board, but no answer was returned. I would ask the managers of this society, how they can expect that the chief end of their institution should be fulfilled, namely, “*To make the way more passable to what remains unrevealed,*” if they disdain the offerings of thinking men? It is probably owing to similar neglects, that the *Philosophical Transactions* have been of little importance for many years past.

This concerns the reader, and I entreat him to look into Dr. Hutton’s celebrated *Mathematical Dictionary*, a work with which he cannot dispense in his library, and under the article “Royal Society,” he will find that learned man’s opinion on the present state of the society; and will be convinced, that the preceding remarks have been made only with a view to the public welfare.

Now, little book! go forth into the world without fear or apprehension, for thou art engaged in the cause of truth. Present thyself to those whose lamps are always burning. Enter into the study and retired room; but tarry not among those archives where the works of the renowned Abraham Sharp are mouldering in obscurity. If thou visitest the colleges, perhaps thou mayest be greeted; but if a more retired life should please thee, thou mayest seek the parterres of Argyle, and perhaps be welcomed by some gardener’s boy; or, perchance, some shepherd may take thee to those lofty fells where thy master made thee, and place thee on the honied ling while his flock is feeding.

Be not ambitious, and farewell!

APPENDIX.

A METHOD OF AFFORDING SOME FACILITY TO THE RESOLUTION OF QUADRATIC EQUATIONS.

THE resolution of quadratics being perfect, all that is attempted here is, to afford some convenience in the operation. By freeing the terms containing the unknown quantities from fractional or high coefficients; such facility will be obtained. This may be done in the following manner :

	Let $ax^2 + bx = c$
putting $y = ax$	
	$y^2 + by = ac$
putting $z = \frac{y}{b}$	
	$z^2 + z = \frac{ac}{b^2}$
and consequently $\frac{bz}{a} = x$	

Ex.

Let $4x^2 + 3x = 22$
$y^2 + 3y = 88$
$z^2 + z = \frac{88}{9}$
$z = \frac{8}{3} \qquad x = \frac{bz}{a} = 2$

Equations of the preceding form have only one real and affirmative root; and therefore there can be no other number than 2 that can answer the conditions of the equation $4x^2 + 3x = 22$.

	Let $ax^2 - bx = c$
putting $y = ax$	
	$y^2 - by = ac$
putting $z = \frac{y}{b}$	
	$z^2 - z = \frac{ac}{b^2}$
and, consequently, $\frac{bz}{a} = x$.	

Ex.

$$\text{Let } 4x^2 - 3x = 370$$

$$y^2 - 3y = 1480$$

$$z^2 - z = \frac{1480}{9}$$

$$z = \frac{40}{3}$$

$$x = \frac{bz}{a} = 10.$$

Equations of the preceding form have only one real and affirmative root; and, consequently, there can be no other number than 10 that can answer the conditions of the equation $4x^2 - 3x = 370$.

$$\text{Let } ax - bx^2 = c$$

$$\text{putting } y = bx$$

$$ay - y^2 = bc$$

$$\text{putting } z = \frac{y}{a}$$

$$z - z^2 = \frac{bc}{a^2}$$

$$\text{and, consequently, } \frac{az}{b} = x$$

Ex.

$$\text{Let } 10x - 2x^2 = 12$$

$$10y - y^2 = 24$$

$$z - z^2 = .24$$

$$z = .4$$

$$x = \frac{az}{b} = 2.$$

But equations of this form, when possible, have always two real and affirmative roots, and their sum is equal to the coefficient of the term which contains the unknown quantity, when the coefficient of the term containing the square of the unknown quantity is unity; but the coefficient of z^2 is 1, therefore another value of z is .6; and, consequently, another value of x is $\frac{10 \times .6}{2} = 3$.

Ex.

$$\text{Let } 40x - 3x^2 = 125$$

$$z - z^2 = \frac{375}{1600} = \frac{15}{64}$$

$$z = \frac{3}{8}, \quad x = \frac{40z}{3} = 5$$

$$z = \frac{5}{8}, \quad x = \frac{40z}{3} = \frac{200}{24} = \frac{25}{3}$$

Since equations of the three forms, having coefficients of any magnitude, may be thus easily converted into other equations, having unity for the coefficient of each term, the three following known rules for their solution, may be applied, in order to find the value of z , by which means some length of operation may be frequently avoided:

When the form is $az - z^2 = c$,
$$z = \frac{a + \sqrt{a^2 - 4c}}{2}$$

When the form is $z^2 + az = c$,
$$z = \frac{\sqrt{a^2 + 4c}}{2} - \frac{a}{2}$$

When the form is $z^2 - az = c$,
$$z = \frac{a + \sqrt{a^2 + 4c}}{2}$$

If the reader will solve the following equations in the usual manner, and also by the method now given, he will be enabled to judge whether the proposed convenience has been afforded or not.

Ex.

$$\text{Let } 3x^2 + 5x = 182$$

$$y^2 + 5y = 546$$

$$y = \frac{\sqrt{25 + 2184}}{2} - 2.5 = 21$$

$$x = \frac{21}{3} = 7.$$

Ex.

$$\text{Let } \frac{x^2}{2} - \frac{x}{3} = 22\frac{1}{6}$$

$$y^2 - \frac{y}{3} = 11\frac{1}{12}$$

$$y = \frac{\frac{1}{3} + \sqrt{\frac{1}{9} + \frac{53}{12}}}{2} = \frac{1}{6} + \frac{20}{6} = 3.5$$

$$x = 2y = 7.$$

Ex.

$$\text{Let } \frac{x^2}{6} + \frac{x}{10} = 25.2$$

$$y^2 + \frac{y}{10} = \frac{25.2}{6}$$

$$z^2 + z = 420$$

$$z=20, y=\frac{z}{10}=2, x=6y=12.$$

Ex.

$$\text{Let } \frac{3x^2}{5} + \frac{x}{4} = 62.5$$

$$y^2 + \frac{y}{4} = 37.5$$

$$z^2 + z = 600$$

$$z=24, y=\frac{z}{4}=6, x=\frac{5y}{3}=10.$$

Ex.

$$\text{Let } \frac{17x^2}{4} - 6x = 12$$

$$y^2 - 6y = 51$$

$$y=10.74596+, x=\frac{4y}{17}=2.528+.$$

Ex.

$$\text{Let } \frac{6x^2}{10} - 5x = 10$$

$$y^2 - 5y = 6$$

$$y=6, x=\frac{10y}{6}=10.$$

Ex.

$$\text{Let } \frac{4\frac{3}{8}x^2}{2\frac{1}{2}} + \frac{x}{2} = 30$$

$$y^2 + \frac{y}{2} = 52.5$$

$$z^2 + z = 210$$

$$z=14, y=\frac{z}{2}=7, x=\frac{2\frac{1}{2}}{4\frac{3}{8}} \times 7=4.$$

Ex.

$$\text{Let } \frac{2x^2}{3} + \frac{3x}{2} = 10.5$$

$$y^2 + \frac{3y}{2} = 7$$

$$z^2 + z = 7 \div \frac{9}{4} = \frac{28}{9}$$

$$z = \frac{4}{3}, y = bz = 2, x = \frac{3y}{2} = 3.$$

Ex.

$$\text{Let } \frac{3x^2}{4} + x = 1$$

$$y^2 + y = .75$$

$$y = .5, x = \frac{4y}{3} = \frac{2}{3}.$$

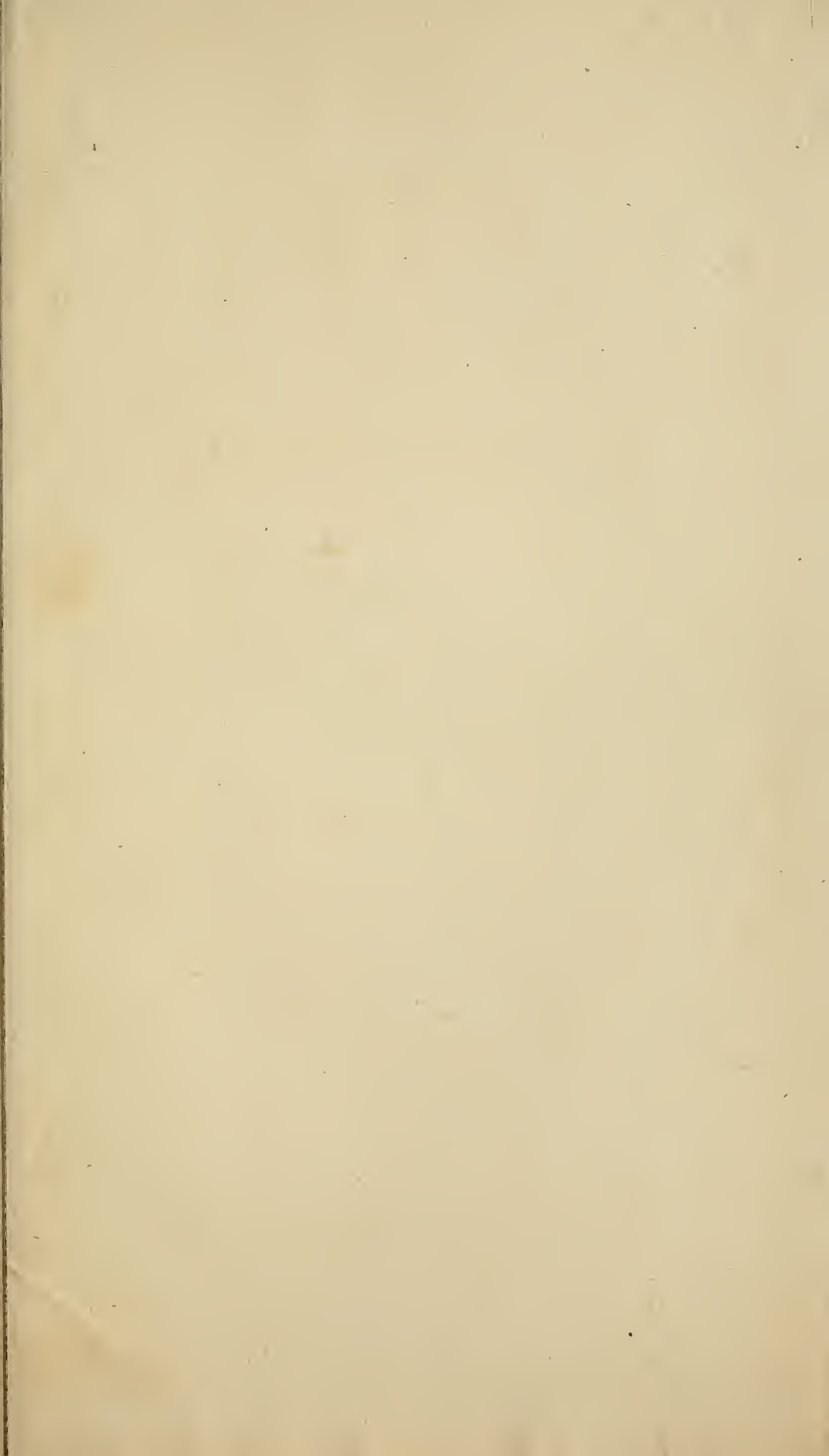
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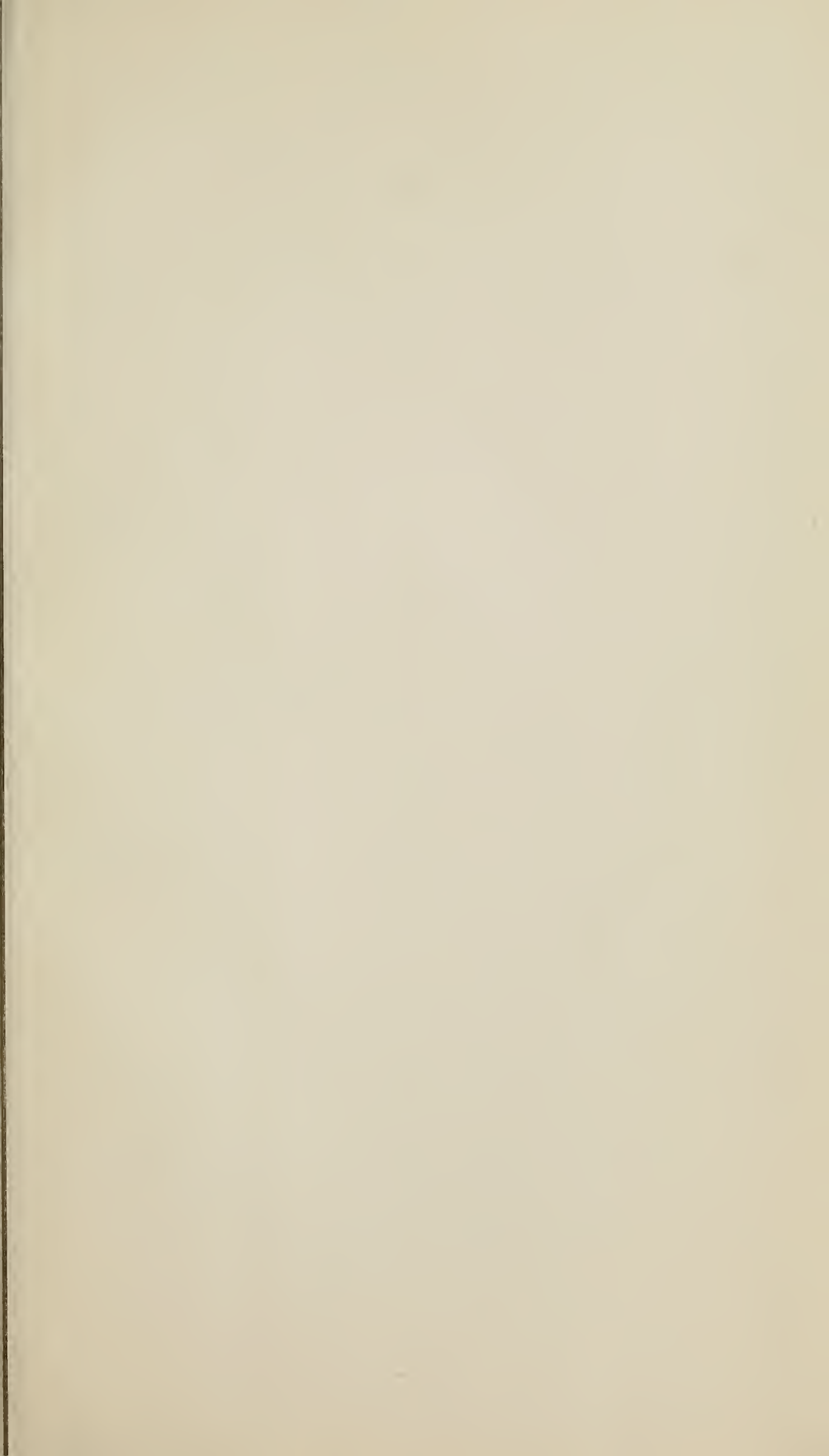
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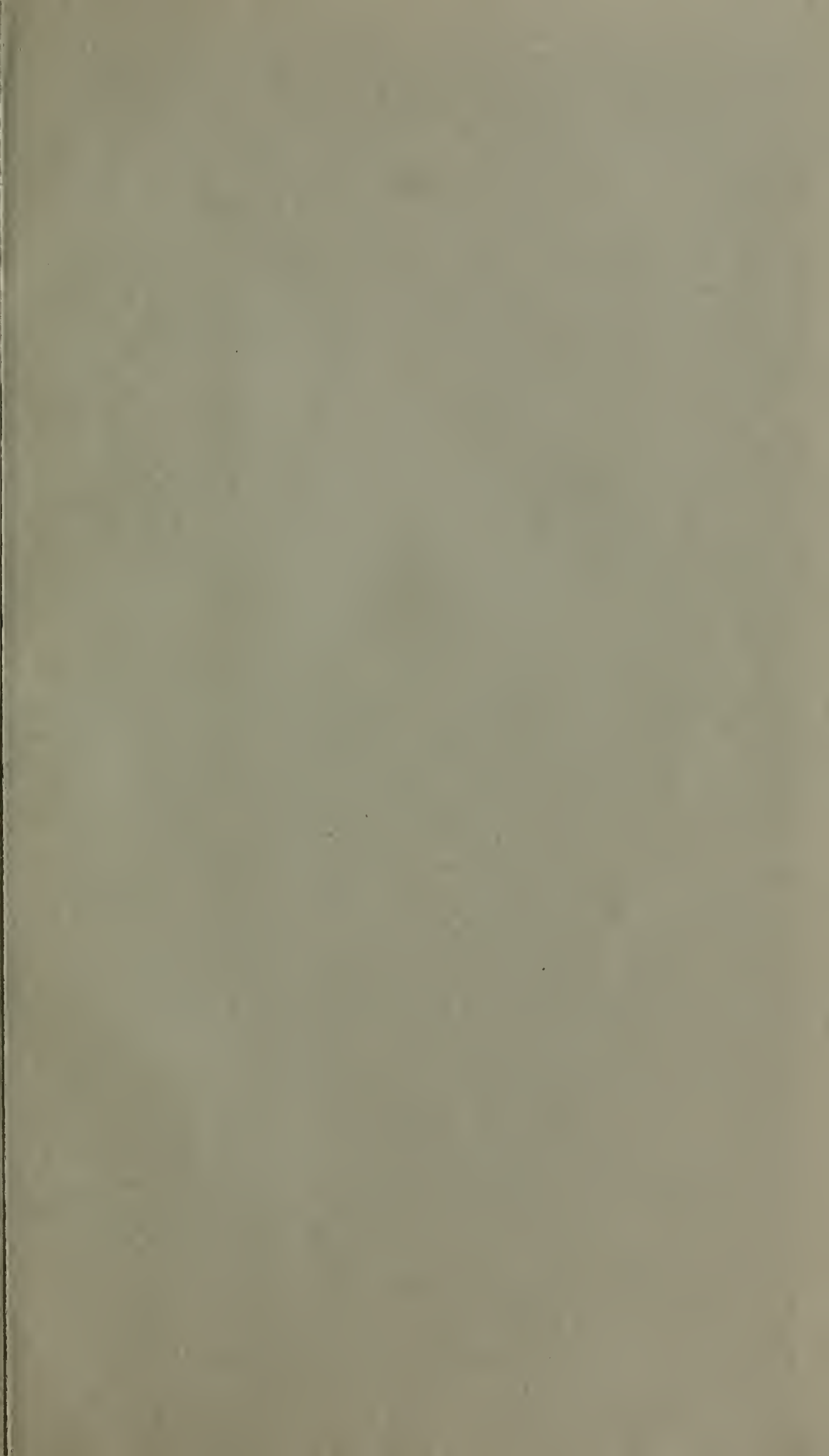
Page 24, line 6, read $(\sqrt{45} + 1) \times x$.

Page 63, line 6 from the bottom, in a few copies only, read $\sqrt{8.5} - \sqrt{8.25}$.









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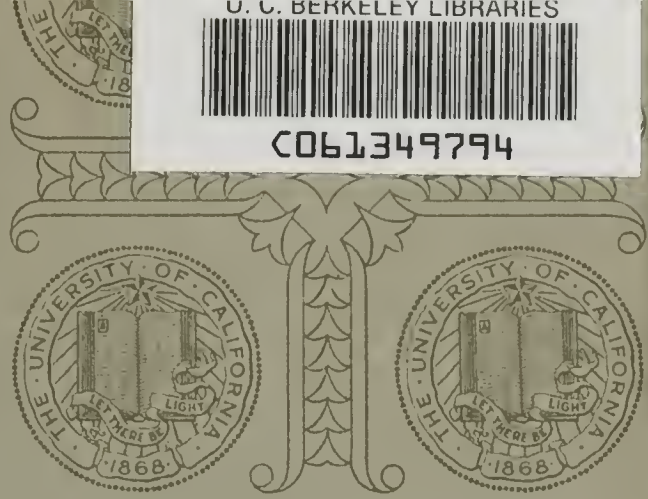
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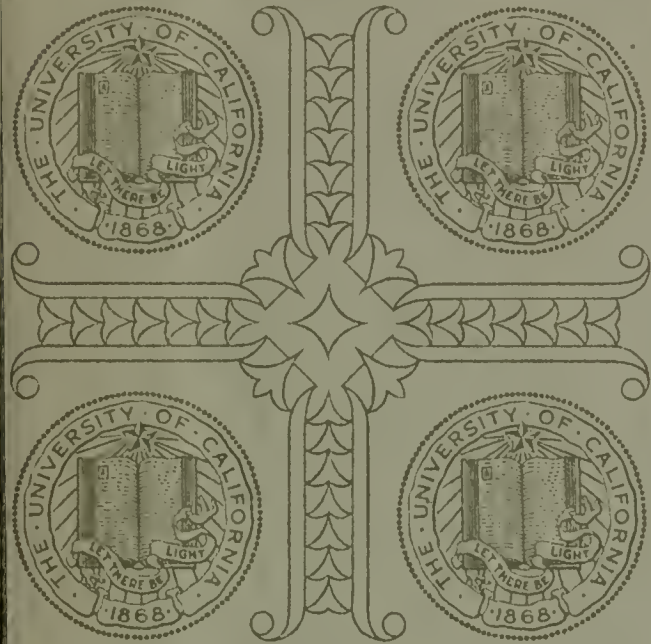


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